TENSOR 2-PRODUCT FOR $\mathfrak{sl}_2$: EXTENSIONS TO THE NEGATIVE HALF

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ABSTRACT. In a recent paper, the author defined an operation of tensor product for a large class of 2-representations of $\mathcal{U}^+$, the positive half of the 2-category associated to $\mathfrak{sl}_2$. In this paper, we prove that the operation extends to give an operation of tensor product for 2-representations of the full 2-category $\mathcal{U}$: when the inputs are 2-representations of the full $\mathcal{U}$, the 2-product is also a 2-representation of the full $\mathcal{U}$. As in the previous paper, the 2-product is given for a simple 2-representation $\mathcal{L}(1)$ and an abelian 2-representation $\mathcal{V}$ taken from the 2-category of algebras.

This is the first construction of an operation of tensor product for higher representations of a full Lie algebra in the abelian setting.

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1. INTRODUCTION

1.1. **Background and motivation.** This paper is the second part in a series by the author, starting with [McM22], about an abelian tensor 2-product operation for 2-representations of Lie algebras. This 2-product is designed with a view to the program by Crane and Frenkel [CF94] seeking a higher representation theory in order to upgrade known $3d$ topological invariants, such as the TQFT of Witten-Reshetikhin-Turaev [Wit89, RT91], to $4d$ invariants.

Prior work in this program involved building categories with Grothendieck groups equal to various representations, including specific tensor products, and these categories have been used to define homological link invariants. This includes early work by Bernstein-Frenkel-Khovanov [BFK99] and later Stroppel and others [Str05, FKS07, MS09, SS15, Sus07] using category $\mathcal{O}$ of $\mathfrak{gl}_n$ for tensor products of simples in type A, and work by Webster [Web17, Web16] using diagrammatic methods for tensor products of simples in other types. We expect these categories to be equivalent (in an appropriate sense) to the tensor 2-products of 2-representations produced by the operation studied in this paper when the factors are simple 2-representations.

The received notion of 2-representation was introduced and developed in [CR08, Lau10, Rou08, KL09, KL11]. A very general definition of tensor 2-product operation for 2-representations of Kac-Moody algebras in the setting of $\mathcal{A}_\infty$-algebras is expected from Rouquier in [Rou]. This general definition does not come with explicit constructions.

In [McM22], the present author gave an explicit abelian model for the tensor 2-product $\mathcal{L}(1) \bigotimes \mathcal{V}$ in the case of $\mathfrak{sl}_2^+$. This is the construction of an algebra, bimodule, and bimodule maps producing a 2-action of $\mathcal{U}^+$, the positive half of the 2-category corresponding to the enveloping algebra of $\mathfrak{sl}_2$. Here $\mathcal{L}(1)$ is a certain simple 2-representation and $\mathcal{V}$ is a given abelian 2-representation taken from the 2-category of algebras and satisfying two additional hypotheses.

A related tensor 2-product for the case of $\mathfrak{gl}(1|1)^+$, which does not involve homotopical complications that are present for $\mathfrak{sl}_2^+$ (due to the absence of endomorphisms $x \in \mathrm{End}(E)$ in the relevant Hecke algebra), was applied by Manion-Rouquier in [MR20] to describe Heegaard-Floer theory for surfaces. Their construction has not been extended to the full $\mathfrak{gl}(1|1)$.

It is not clear whether a 2-representation theory for $\mathfrak{sl}_2^+$ could suffice to build a TQFT, and it is natural to ask whether the construction in [McM22] can be extended to $\mathfrak{sl}_2$. The main result of this paper is a proof that it can indeed be extended. It gives, then, the first case of a 2-product operation in the abelian setting for a *full* Lie algebra or super-Lie algebra, while [McM22] gave an operation for a *half* Lie algebra (in an abelian setting), whereas [MR20] used an operation for a *half super*-Lie algebra (in a dg-setting).

1.2. **Result.** Let $\mathcal{U}$ be the 2-category associated with the enveloping algebra of $\mathfrak{sl}_2$, as given in Rouquier [Rou08, §4.1.3] or Vera [Ver20, §3.2]. Let $\mathcal{U}^+$ be the monoidal category associated to the positive half of the enveloping algebra of $\mathfrak{sl}_2$. As in [McM22, §1.2], we work with 2-representations in the abelian 2-category of algebras, bimodules, and bimodule maps.

Let A be a k-algebra for a field k, let E be an (A, A)-bimodule, and let $x \in \mathrm{End}(E)$, $\tau \in \mathrm{End}(E^2)$ be bimodule endomorphisms satisfying the nil affine

Hecke relations:

$$\begin{gathered}\tau^2 = 0,\\ \tau E \circ E\tau \circ \tau E = E\tau \circ \tau E \circ E\tau, \\ \tau \circ Ex = xE \circ \tau + 1,\ Ex \circ \tau = \tau \circ xE + 1.\end{gathered} \tag{1.1}$$

(The notation xE indicates the endomorphism $x \otimes \mathrm{Id}_E$ in $\mathrm{End}(E^2)$, etc.) The data (A, E, x, τ) determines a 2-representation of $\mathcal{U}^+$.

Now assume that (A, E, x, τ) has a weight decomposition $A = \prod_{\lambda\in\mathbb{Z}} A_\lambda$ (cf. [McM22, §4.3.1]). The data (A, E, x, τ) extends to determine a 2-representation of the full 2-category $\mathcal{U}$ when the functor $E \otimes_A -$ admits a right adjoint functor F (with unit η and counit ε) such that the "commutator" maps ρ_λ (determined by x, τ, η, ε; see §2.2 below) are isomorphisms in each weight $\lambda \in \mathbb{Z}$.

A simple 2-representation $\mathcal{L}(1)$ of $\mathcal{U}$ that categorifies the fundamental representation $L(1)$ of $\mathfrak{sl}_2$ may be given by the following data. Let the k-algebra be $k[y]_{+1} \times k[y]_{-1}$ (decomposed into weight algebras), and the triple be $(k[y], y, 0)$. Let x act by multiplication by y. Let $y \in k[y]_{-1}$ act on $k[y]$ on the right by multiplication, and $y \in k[y]_{+1}$ act by zero; swap them for the left action.

Let $P_n = k[x_1, \dots, x_n]$ be the polynomial algebra. Then P_n acts on E^n with $x_i \in P_n$ acting by the endomorphism $E^{n-i}xE^{i-1}$.

Theorem (Main result)**.** *Suppose (A, E, x, τ) gives the data of a 2-representation $\mathcal{V}$ of $\mathcal{U}^+$ such that $\mathcal{V}$ has a weight decomposition. Define the left-dual (A, A)-bimodule $F = \mathrm{Hom}_A({}_AE, A)$. Suppose E has the following properties:*

- *${}_AE$ is finitely generated and projective, so $(E\otimes_A -, F\otimes_A -)$ is an adjunction where the unit η and counit ε arise from the duality pairing,*
- *E^n is free as a P_n-module,*
- *E and F are locally nilpotent,*
- *The maps ρ_λ determined by the given data are isomorphisms for each $\lambda \in \mathbb{Z}$.*

These properties imply that $(A, E, F, x, \tau, \eta, \varepsilon)$ determines an integrable 2-representation of $\mathcal{U}$.

Now let C be the k-algebra, $\tilde{E}$ the (C, C)-bimodule, and $\tilde{x}$ and $\tilde{\tau}$ the bimodule endomorphisms constructed in [McM22]*. Let $\tilde{F} = \mathrm{Hom}_C({}_C\tilde{E}, C)$. Then:*

- *${}_A\tilde{E}$ is finitely generated and projective, so $(\tilde{E} \otimes_C -, \tilde{F} \otimes_C -)$ is an adjunction with unit $\tilde{\eta}$ and counit $\tilde{\varepsilon}$ arising from the duality pairing,*
- *$\tilde{E}$ and $\tilde{F}$ are locally nilpotent,*
- *The maps $\tilde{\rho}_\lambda$ determined by the given data are isomorphisms, so:*
- *$(C, \tilde{E}, \tilde{F}, \tilde{x}, \tilde{\tau}, \tilde{\eta}, \tilde{\varepsilon})$ determines an integrable 2-representation of $\mathcal{U}$.*

The data $(C, \tilde{E}, \tilde{x}, \tilde{\tau})$ determines a 2-representation of $\mathcal{U}^+$ that we interpreted in [McM22] as the result $\mathcal{L}(1) \circledcirc \mathcal{V}$ of a 2-product operation (with the factors considered as 2-representations of $\mathcal{U}^+$). One reason to interpret the structure in this way was that it results from a categorification of the Hopf coproduct formula. Another reason was that it recovers the expected structures in some known cases. For details, see [McM22] in §1.3, §1.4, as well as in Remark 3.4 about the effect of E' and thus $\tilde{E}$ on the Grothendieck group. Since the additional components $\tilde{F}$, $\tilde{\eta}$, and $\tilde{\varepsilon}$ are fully determined by $(C, \tilde{E}, \tilde{x}, \tilde{\tau})$, in this article we interpret the 2-representation determined by the combined data $(C, \tilde{E}, \tilde{F}, \tilde{x}, \tilde{\tau}, \tilde{\eta}, \tilde{\varepsilon})$ as the result $\mathcal{L}(1) \circledast \mathcal{V}$ of a 2-product operation with the factors considered as 2-representations of $\mathcal{U}$.

We emphasize that for an integrable 2-representation of $\mathcal{U}^+$ given by the data (A, E, x, τ), the fact that the data determines a 2-representation of the full 2-category $\mathcal{U}$ is equivalent to the data having a property: namely that ${}_AE$ is f.g. projective, and the commutator maps ρ_λ determined by the data are isomorphisms. When this holds, then (according to the theorem) the maps $\tilde{\rho}_\lambda$ of the product are also isomorphisms. So the new data $(C, \tilde{E}, \tilde{x}, \tilde{\tau})$ inherits the property of determining an action of the full $\mathcal{U}$.

1.3. **Outline summary.** The paper is organized as follows:

- In §2 we introduce the relevant background theory for extensions of 2-representations of $\mathcal{U}^+$ to the full 2-category $\mathcal{U}$. This section builds on the background theory and definitions of [McM22]. We include a discussion of the adjunction, the commutator maps ρ_λ, and the condition of integrability as it relates to our product construction.
- In §3 we define and study important bimodules, giving concrete algebraic models for them in the manner of §3.2 of [McM22].
- In §4 we consider the left dual to $\tilde{E}$, namely $\tilde{F} = \mathrm{Hom}_C({}_C\tilde{E}, C)$, and we show how to describe it concretely by using the B side of the equivalence described in §3.3.2 of [McM22].
- In §5.1 we study the tensor products $\tilde{E} \otimes_C \tilde{E}$ and $\tilde{E} \otimes_C \tilde{F}$ and $\tilde{F} \otimes_C \tilde{E}$, and describe their structure as $(A[y], A[y])$-bimodules. In §5.2 we compute explicit formulas for $\tilde{\rho}_\lambda$ in terms of the structures found in §5.1. In §5.3 we use the formulas from §5.2 to show that each $\tilde{\rho}_\lambda$ is an isomorphism.

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2. Background: extending $\mathcal{U}^+$ actions to $\mathcal{U}$ actions

2.1. **2-Representations of $\mathcal{U}$.** We begin with a description of a 2-representation of the full 2-category $\mathcal{U}$ associated to the Lie algebra $\mathfrak{sl}_2$. The 2-category $\mathcal{U}$ that we mean is defined in §4.1.3 of [Rou08], but with τ replaced by $-\tau$ in the Hecke relations. We do not repeat that definition here since we work with the concrete data of 2-representations and not with the 2-category $\mathcal{U}$ itself.

In [McM22], a 2-representation was defined as a strict monoidal functor from $\mathcal{U}^+$ to a monoidal category of the form $\mathsf{Bim}_k(A)$, which is defined for a k-algebra A as follows: the objects of $\mathsf{Bim}_k(A)$ are (A, A)-bimodules, and the morphisms of $\mathsf{Bim}_k(A)$ are bimodule maps. The monoidal structure on $\mathsf{Bim}_k(A)$ is given by tensor product of bimodules over A. This monoidal category $\mathsf{Bim}_k(A)$ may also be interpreted as a 2-category with a single object A, where the 1-morphisms are given by tensor product with (A, A)-bimodules, and the 2-morphisms are bimodule maps.

A 2-representation of the full $\mathcal{U}$ is defined in terms of weights (see Def. 4.25 of [McM22]). When A is provided with a weight decomposition $A = \prod_{\lambda \in \mathbb{Z}} A_\lambda$, then the 2-category $\mathsf{Bim}_k(A)$ with single object A may be expanded to a 2-category with objects given by the weight algebras A_λ, morphisms given by (A_μ, A_λ)-bimodules, and 2-morphisms given by bimodule maps. With this interpretation, a 2-representation of $\mathcal{U}$ may be described as a strict 2-functor $\mathcal{U} \to \mathsf{Bim}_k(A)$ given on objects by $\mathbf{1}_\lambda \mapsto A_\lambda$.

According to Prop. 2.4 of [McM22], a 2-representation of $\mathcal{U}^+$ in $\mathsf{Bim}_k(A)$ is equivalent to the data of a k-algebra A together with a bimodule ${}_AE_A$ and bimodule maps $x \in \mathrm{End}(E)$, $\tau \in \mathrm{End}(E^2)$ satisfying relations (1.1). This paper will rely on the following analogue of that proposition:

Proposition 2.1. *The data of a 2-representation $\mathcal{U} \to \mathsf{Bim}_k(A)$ for a k-algebra $A = \prod_{\lambda \in \mathbb{Z}} A_\lambda$ consists of bimodules ${}_AE_A$, ${}_AF_A$ (having weights $+2$ and -2), the unit η and counit ε of an adjunction (E, F), and bimodule maps $x \in \mathrm{End}(E)$, $\tau \in \mathrm{End}(E^2)$ that satisfy relations (1.1), all such that ρ_λ (defined below in terms of x, τ, η, ε) is an isomorphism for each λ.*

(Bimodules E, F are said to have weight $+2$ and -2, respectively, when $e_jEe_i = \delta_{i+2,j} \cdot e_{i+2}Ee_i$ and $e_jFe_i = \delta_{i-2,j} \cdot e_{i-2}Fe_i$.)

In this paper, a symbol $\mathcal{V}$ is used sometimes to denote a 2-representation of $\mathcal{U}^+$, and sometimes to denote the extension of the former to a 2-representation of $\mathcal{U}$. This is an abuse of notation because the first $\mathcal{V}$ is a monoidal category, and the second $\mathcal{V}$ is a 2-category. This abuse is justifiable when both types of category are determined by the same data.

2.2. **Commutator morphisms.** Here we define the commutator morphisms. Assume we are given the data of a k-algebra A, bimodules ${}_AE_A$, ${}_AF_A$ which determine endofunctors of A-mod by tensor product on the left, the unit η and counit ε of an adjunction (E, F), and endofunctors x and τ satisfying (1.1). Assume that A has a weight decomposition $A = \prod_{\lambda \in \mathbb{Z}} A_\lambda$, and E and F have weights $+2$ and -2. Let us use the notation $E_\lambda = E \cdot A_\lambda$ and ${}_\mu E_\lambda = A_\mu \cdot E \cdot A_\lambda$, so $E = \bigoplus_{\mu,\lambda} {}_\mu E_\lambda$. In this paper we also use a convention that '$\oplus$' and '$\sum$' denote the components of a map to and from a direct sum, respectively.

We define $\sigma : EF \to FE$ by:

$$\sigma = (FE\varepsilon) \circ (F\tau F) \circ (\eta EF) : EF \to FE.$$

For $\lambda \in \mathbb{Z}_{\geqslant 0}$ we define:

$$\rho_\lambda = \sigma \oplus \bigoplus_{i=0}^{\lambda-1} \varepsilon \circ x^i F : EF_\lambda \to FE_\lambda \oplus A_\lambda^{\oplus \lambda}, \tag{2.1}$$

and for $\lambda \in \mathbb{Z}_{\leqslant 0}$:

$$\rho_\lambda = \left(\sigma, \sum_{i=0}^{-\lambda-1} Fx^i \circ \eta \right) : EF_\lambda \oplus A_\lambda^{\oplus -\lambda} \to FE_\lambda. \tag{2.2}$$

(The summation terms are neglected when $\lambda = 0$.)

2.3. **Conventions.** We adopt the conventions of [McM22], so the reader may consult §2.3 of that text for additional details. Assume we are given data (A, E, x, τ) determining a 2-representation $\mathcal{V}$ of $\mathcal{U}^+$. Assume that ${}_AE$ is f.g. projective and that E^n is free as a P_n-module.

The construction of the product $\mathcal{L}(1) \circledast \mathcal{V}$ in [McM22] makes use of the $(A[y], A[y])$-bimodule $E[y]$, and the endomorphism $x - y \in \mathrm{End}(E[y])$. Write E_y for the quotient $E[y]/(x-y)E[y]$, and $\pi : E[y] \to E_y$ for its projection.

Concatenation of the symbols for bimodules indicates tensor product over some algebra that is determined by context. Sometimes this algebra could be either A or $A[y]$, so we stipulate that if the expression for a bimodule contains 'y', it will be understood as an $(A[y], A[y])$-bimodule, and if the expression lacks 'y', it will be understood as an A-module. We suppress isomorphisms such as:

$$E[y]E_y = E[y] \otimes_{A[y]} E_y \xrightarrow{\sim} E \otimes_A E_y = EE_y.$$

Extend x to an element of $\mathrm{End}(E[y])$ by $x : ey^n \mapsto x(e)y^n$ and τ to $\mathrm{End}(E^2[y])$ by $\tau : eey^n \mapsto \tau(ee)y^n$. When writing formulas for morphisms we often write an arbitrary element of $E[y]$ with the single letter 'e' and an arbitrary element of $E^2[y]$ with the doubled symbol 'ee' (which is not assumed to be a simple tensor).

We make use of the notation $y_i = x_i - y$. Here y_i indicates $\left(E^j x E^{i-1} - y\right)$ for some j, and context will determine the value of j.

As in §2.3 of [McM22], let $s \in \mathrm{End}(E^2)$ be the bimodule map given by $s = \tau \circ (x_1 - x_2) - \mathrm{Id}$, and extended to $E^2[y]$ as x and τ are extended. Note that s descends to define maps of $(A[y], A[y])$-bimodules $s : E_yE \to EE_y$ and $s : EE_y \to E_yE$ such that s^2 descends to Id.

2.4. **Adding a dual.** Every bimodule ${}_AE_A$ has left- and right-dual bimodules,

$$\begin{aligned} {}^{\vee}E &= \mathrm{Hom}_A({}_AE, A), \\ E^{\vee} &= \mathrm{Hom}_A(E_A, A), \end{aligned}$$

respectively.

Now, when ${}_AE$ is f.g. projective, the canonical morphism ${}^{\vee}E \otimes_A E \to \mathrm{Hom}_A({}_AE, E)$ is an isomorphism of bimodules. More generally, the canonical morphism of functors ${}^{\vee}E \otimes_A - \to \mathrm{Hom}_A({}_AE, -)$ is an isomorphism. In this situation, the endofunctor ${}^{\vee}E \otimes_A -$ of the category A-mod is right adjoint to the endofunctor $E \otimes_A -$ of the same category. The triple $({}^{\vee}E, \eta, \varepsilon)$ gives the right-dual object for E in the monoidal category $\mathsf{Bim}_k(A)$. Here $\varepsilon : E \otimes_A {}^{\vee}E \to A$ is given by evaluation, and $\eta : A \to {}^{\vee}E \otimes_A E$ is given via the isomorphism ${}^{\vee}E \otimes_A E \xrightarrow{\sim} \mathrm{Hom}_A({}_AE, E)$ by the right A-action whereby $\eta(a) : e \mapsto e.a$. (Note that we say ${}^{\vee}E$ is the *left*-dual bimodule, even though it gives the *right*-dual object.)

Conversely, assume that $(E \otimes_A -, {}^{\vee}E \otimes_A -)$ is an adjoint pair for some bimodule ${}_AE_A$. The adjunction gives equivalences of functors:

$$\mathrm{Hom}_A({}_AE, -) \cong \mathrm{Hom}_A({}_AA, {}^{\vee}E \otimes_A -) \cong {}^{\vee}E \otimes_A -,$$

so all three are both right- and left-exact functors. So ${}_AE$ is projective. Furthermore, these functors commute with infinite direct sums, so ${}_AE$ is finitely generated as well.

In this paper we consider 2-representations for which the image of F in $\mathsf{Bim}_k(A)$, i.e. the bimodule ${}_AF_A$, is identically the left-dual bimodule ${}^\vee E$. There is no loss of generality because any 2-representation of $\mathcal{U}$ in $\mathsf{Bim}_k(A)$ is equivalent to one of these. (For any 2-representation in $\mathsf{Bim}_k(A)$, the endofunctor ${}_AF \otimes_A -$ of A-mod is right adjoint to ${}_AE \otimes_A -$, and is therefore unique up to unique isomorphism.) A 2-representation of $\mathcal{U}$ given by the data $(A, E, F, x, \tau, \eta, \varepsilon)$ in $\mathsf{Bim}_k(A)$ is said to *extend* a 2-representation (A, E, x, τ) of $\mathcal{U}^+$ when $F = {}^\vee E$ and η, ε arise from the duality.

It was a hypothesis of the main theorem of [McM22] that ${}_AE$ is f.g. projective. This condition was needed in order to show that $E'X$ was a perfect complex (for example). In light of the above, we see that the existence of an extension of the 2-representation of $\mathcal{U}^+$ to a 2-representation of $\mathcal{U}$ in $\mathsf{Bim}_k(A)$ also necessitates that hypothesis.

The following lemma is a consequence of the foregoing discussion.

Lemma 2.2. *Suppose the data (A, E, x, τ) determines a 2-representation of $\mathcal{U}^+$ in $\mathsf{Bim}_k(A)$ having a weight decomposition. This data extends to determine a 2-representation of $\mathcal{U}$, with $F = {}^\vee E$, if and only if ${}_AE$ is f.g. projective and the commutator morphisms ρ_λ (determined by x, τ, η, ε) are isomorphisms.*

In [McM22] the author defined the data $(C, \tilde{E}, \tilde{x}, \tilde{\tau})$ of a product 2-representation of $\mathcal{U}^+$ in terms of given data (A, E, x, τ) satisfying some conditions. In that paper it was seen that ${}_C\tilde{E}$ is f.g. projective, and it follows that $\tilde{F} = {}^\vee\tilde{E}$ is right adjoint to $\tilde{E}$. In this paper we aim to show that $(C, \tilde{E}, \tilde{F}, \tilde{x}, \tilde{\tau}, \tilde{\eta}, \tilde{\varepsilon})$ determines a 2-representation of $\mathcal{U}$. Our argument uses the above Lemma: it will suffice to show that the commutator morphisms $\tilde{\rho}_\lambda$ are isomorphisms.

2.5. **Integrability.** In the literature, a 2-representation is typically defined in terms of weight categories $\mathcal{C}_\lambda$ and functors E and F between them, whereas we have framed our results entirely in terms of bimodules E and F. One reason for this is that a certain pair of bimodules may determine several functors (by the operation of tensoring on the left) that act on several reasonable categories of modules. The most important ones are A-mod and A-proj.

The distinction between A-mod and A-proj interacts with our results and the hypothesis of integrability in an interesting way. This interaction is mediated by the property of "second adjunction" that a 2-representation of $\mathcal{U}$ may possess. We explain this next. Note that some authors include the second adjunction in their definition of a 2-representation, and for them, this discussion will be of minor significance. It may be interesting for them to observe, though, that in our construction of tensor product, the hypothesis of integrability passes from the factors to the product quite easily, while it is not clear that a second adjunction alone passes from the factors to the product at all.

Every 2-representation of $\mathcal{U}$ given with functors E and F comes with one adjunction (E, F), and with the data of a "candidate" unit and counit pair for a second adjunction (F, E). When the 2-representation acts on a category

A-mod and E and F are given by tensoring with bimodules, the first adjunction implies that ${}_AE$ is f.g. projective. In this case, the upper half $\mathcal{U}^+$ also acts on the smaller category A-proj. If the 2-representation is assumed to be integrable, and the full $\mathcal{U}$ acts, i.e. the ρ_λ are isomorphisms, then by Theorem 5.16 of [Rou08] the given candidates do provide a second adjunction (F, E). This adjunction implies that ${}_AF$ is also f.g. projective, and now the full $\mathcal{U}$ action may be restricted to A-proj.

Given only the first adjunction with an action of $\mathcal{U}^+$, so ${}_AE$ is f.g. projective, together with the hypothesis that E^n is free over P_n, we can form the 2-representation of $\mathcal{U}^+$ called $\mathcal{L}(1) \bigotimes \mathcal{V}$ in [McM22]. In that paper it was shown that ${}_C\tilde{E}$ is f.g. projective, so it may be interpreted either in an action on C-mod or else in an action restricted to C-proj. Given also a second adjunction $({}^\vee E, E)$ determining an action of the full $\mathcal{U}$, we know that $\mathcal{U}$ acts on A-proj through E and ${}^\vee E$ in the 2-representation $\mathcal{V}$, but we are not (currently) able to show from this alone that $\mathcal{U}$ acts on C-proj through $\tilde{E}$ and ${}^\vee\tilde{E}$, since we do not know that ${}^\vee\tilde{E}$ is f.g. projective.

Given the first adjunction $(E, {}^\vee E)$ and also the hypothesis of integrability of an action of the full $\mathcal{U}$, we know that there is a second adjunction $({}^\vee E, E)$. Now the hypothesis of integrability itself passes to the product bimodule $\tilde{E}$. (Prop. 4.24 of [McM22].) Given that we can also show that the product maps $\tilde{\rho}_\lambda$ are isomorphisms (the main effort of this paper), so we have an action of the full $\mathcal{U}$ on C-mod, it follows from integrability that there is a second adjunction $({}^\vee\tilde{E}, \tilde{E})$ for the product. This implies, in turn, that ${}_C{}^\vee\tilde{E}$ is f.g. projective and that the full $\mathcal{U}$ action may be restricted to the category C-proj.

To summarize, second adjunctions enable restriction of the full $\mathcal{U}$ action to the subcategories A-proj and C-proj. The existence of a second adjunction $({}^\vee E, E)$ in $\mathcal{V}$ is *not enough* (with the arguments below) to guarantee a second adjunction $({}^\vee\tilde{E}, \tilde{E})$ in $\mathcal{L}(1) \bigotimes \mathcal{V}$. But *integrability* of $\mathcal{V}$ is enough to guarantee *integrability* of $\mathcal{L}(1) \bigotimes \mathcal{V}$, as well as to give *both* second adjunctions $({}^\vee E, E)$ and $({}^\vee\tilde{E}, \tilde{E})$.

2.6. **Background: 2-product for $\mathcal{U}^+$.** We recall some definitions and results from [McM22]. The reader is encouraged to review that paper and to consult it for additional details and conventions.

Definition 2.3 (Def. 3.1 of [McM22])**.** Let B be the k-algebra:

$$B = \begin{pmatrix} A[y] & E_y \\ 0 & A[y] \end{pmatrix}.$$

The algebra structure of B is given by matrix multiplication, where products involving generators in $[B]_{12}$ are defined using the bimodule structure of E_y.

Modules over B are naturally written in terms of components. A left B-module is given by a pair $\left(\begin{smallmatrix} M_1 \\ M_2 \end{smallmatrix}\right)$ of left $A[y]$-modules, together with a morphism $\alpha : E_y \otimes_{A[y]} M_2 \to M_1$ of left $A[y]$-modules specifying the action of E_y generators; analogously a right B-module is given by a pair $\left(\begin{smallmatrix} N_1 & N_2 \end{smallmatrix}\right)$ and morphism $\beta : N_1 \otimes_{A[y]} E_y \to N_2$.

A bimodule consists of a 2×2 matrix with additional data. The direct sum of coefficients in the top row of such a matrix gives the top component of the pair corresponding to the left-module structure, and the bottom row gives the bottom component of the pair; similarly the columns give the components of the right-module structure. The additional data consists of α determining 'vertical' maps and β giving 'horizontal' maps. A matrix together with maps α and β determines a bimodule only if the left and right actions of E_y specified by α and β commute. (In this situation the vertical and horizontal maps respect the decompositions into horizontal and vertical components, respectively.)

A complex of left B-modules is equivalent to a pair of complexes of $A[y]$-modules and a map of complexes α; analogously for right B-modules and for bimodules. Complexes in this paper have a cohomological grading.

Definition 2.4 (Def. 3.2 of [McM22])**.** Let E' be the complex of (B, B)-bimodules that is nonzero in degrees 0 and 1, where it is given by:

$$E'_0 = \begin{pmatrix} E[y] & E[y]E_y \\ 0 & E[y] \end{pmatrix}, \; E'_1 = \begin{pmatrix} E_y & E_yE_y \\ A[y] & E_y \end{pmatrix}.$$

The left action of a generator in $E_y \subset B$ is specified on vertical columns of E'_0 by the maps $0 : E_y \otimes 0 \to E[y]$ and $s : E_yE[y] \to E[y]E_y$. The left action on E'_1 is specified by the identity map on vertical columns. The right action on E'_0 is specified by the identity on the top row, and 0 on the bottom row. The right action on E'_1 is specified by the identity on both rows. The differential $E'_0 \to E'_1$ is given *componentwise* by $\left(\begin{smallmatrix} \pi & \pi E_y \\ 0 & \pi \end{smallmatrix}\right)$.

Lemma 2.5 (Lemma 3.3 of [McM22])**.** *Let $M = \left(\left(\begin{smallmatrix} M_1 \\ M_2 \end{smallmatrix}\right), \alpha\right)$ be a complex of left B-modules (written as a pair of complexes), where $\alpha : E_y \otimes_{A[y]} M_2 \to M_1$ specifies the action for generators in E_y. The functor $E' \otimes_B -$ on M may be given by:*

$$\left(\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}, \alpha\right) \xmapsto{E'} \left(\begin{pmatrix} E[y]M_1 \overset{\pi M_1}{\curvearrowright} \oplus E_yM_1[-1] \\ E[y]M_2 \overset{\alpha\circ\pi M_2}{\curvearrowright} \oplus M_1[-1] \end{pmatrix}, \begin{pmatrix} E[y]\alpha \circ sM_2 & 0 \\ 0 & Id_{E_yM_1} \end{pmatrix}\right).$$

Here the top and bottom rows express cocones of the maps πM_1 and $\alpha \circ \pi M_2$.

Definition 2.6 (Def. 3.5 of [McM22])**.** Let X be the following complex of B-modules:

$$X = X_1 \oplus X_2; \quad X_1 = \begin{pmatrix} A[y] \\ 0 \end{pmatrix}, \quad X_2 = E'X_1 = \begin{pmatrix} E[y] & \xrightarrow{\pi} & E_y \\ 0 & \longrightarrow & A[y] \end{pmatrix},$$

where X_1 lies in degree 0 and X_2 in degrees 0 and 1. The E_y action on X_2 is given by $E_y \otimes_{A[y]} A[y] \xrightarrow{\sim} E_y$, $e \otimes 1 \mapsto e$.

Proposition 2.7 (Prop. 3.6 of [McM22])**.** *The complex X is strictly perfect and generates per B, the full subcategory of $D^b(B)$ of complexes quasi-isomorphic to strictly perfect complexes.*

Next we recall an important series of bimodules introduced in [McM22]:

Definition 2.8 (Def. 3.16 of [McM22])**.** Let G_n denote $\mathrm{Hom}_{K^b(B)}(X_2, E'^n X_1)$.

Note that the quotient projection to the derived category is an isomorphism $G_n \xrightarrow{\sim} \mathrm{Hom}_{D^b(B)}(X_2, E'^n X_1)$ because X_2 is strictly perfect. Note also that G_1 has an algebra structure given by composition of endomorphisms.

Proposition 2.9 (Props. 3.18, 3.20, and 3.22 together with 3.27 and 3.28 of [McM22])**.** *There are isomorphisms of $(A[y], A[y])$-bimodules $\bar{G}_1 \xrightarrow{\sim} G_1$, $\bar{G}_2 \xrightarrow{\sim} G_2$, $\bar{G}_3 \xrightarrow{\sim} G_3$, where:*

$$\begin{aligned} \bar{G}_1 = \Bigg\langle (\theta, \varphi) \in A^{\mathsf{op}}[y] \oplus \mathrm{End}_A({}_AE)[y] \Bigg| \\ \varphi = {\_}.\theta + y_1\varphi_1 \\ \textit{for some } \varphi_1 \in \mathrm{End}_A({}_AE)[y] \Bigg\rangle, \end{aligned}$$

$$\begin{aligned} \bar{G}_2 = \Bigg\langle (e_1, e_2, \xi) \in E[y]^{\oplus 2} \oplus \mathrm{Hom}_A({}_AE, E^2)[y] \Bigg| \\ e_1 - e_2 &= y_1 e' \\ \xi &= {\_} \otimes e_1 + y_2\xi_1 \\ \xi_1 &= \tau({\_} \otimes e_2) + y_1\xi' \\ \textit{for some } e' \in E[y] \textit{ and } \xi' \in \mathrm{Hom}_A({}_AE, E^2)[y] \Bigg\rangle, \end{aligned}$$

$$\begin{aligned} \bar{G}_3 = \Bigg\langle (ee_1, ee_2, ee_3, \chi) \in E^2[y]^{\oplus 3} \oplus \mathrm{Hom}_A({}_AE, E^3)[y] \Bigg| \\ ee_1 - ee_2 &= y_2 ee' \\ ee_3 - ee_2 &= y_1 ee'' \\ \tau y_1(ee_3) - ee_1 &= y_1 ee''', \\ \chi &= {\_} \otimes ee_1 + y_3\chi_1 \\ \chi_1 &= \tau E({\_} \otimes ee_2) + y_2\chi_1' \\ \chi_1' &= E\tau \circ \tau E({\_} \otimes ee_3) + y_1\chi'' \\ \textit{for some } ee^{(k)} \in E^2[y] \textit{ and } \chi'' \in \mathrm{Hom}_A({}_AE, E^3)[y] \Bigg\rangle. \end{aligned}$$

Here ${\_}.\theta \in \mathrm{End}_A({}_AE)[y]$ is the morphism sending e to $e.\theta$, and ${\_} \otimes e_1 \in \mathrm{Hom}_A({}_AE, E^2)[y]$ sends e to $e \otimes e_1$, and ${\_} \otimes ee_1$ sends e to $e \otimes ee_1$. Note that e', ξ_1, and ξ' are uniquely determined by (e_1, e_2, ξ), and ee', ee'', ee''' and χ_1, χ_1', χ'' are uniquely determined by (ee_1, ee_2, ee_3, χ).

We rely on this proposition in what follows and do not distinguish between G_i and $\bar{G}_i$ for $i = 1, 2, 3$. For example, we may write $(\theta, \varphi) \in G_1$. (We also write (θ, φ) for the element of G_1^{op}.) The interpretations of elements (θ, φ) etc. as explicit homomorphisms of complexes are given by Props. 3.18, 3.20, and 3.22 of [McM22].

Also recall the two complexes of B-modules:

Definition 2.10. Let $R, X'_2 \in B$-cplx be given by:

$$R = \begin{pmatrix} E^2[y] \xrightarrow{\left(\begin{smallmatrix} \pi E \\ \pi E \circ \tau \end{smallmatrix}\right)} E_y E \oplus E_y E \\ 0 \to E[y] \oplus E[y] \end{pmatrix},$$

$$X'_2 = \begin{pmatrix} \tau y_1 E^2[y] & \xrightarrow{\pi E} & E_y E \\ 0 & \longrightarrow & E[y] \end{pmatrix},$$

both lying in degrees 0 and 1 (cohomological grading), and the E_y action on R specified by 0 and the canonical map

$$E_y \otimes (E[y] \oplus E[y]) \to E_y E \oplus E_y E,$$

and on X'_2 specified by 0 and $\mathrm{Id}_{E_y E[y]}$.

Now recall the following three lemmas:

Lemma 2.11 (Lem. 3.11 of [McM22])**.** *We have that X'_2 is a finite direct sum of summands of X.*

The nil-affine Hecke algebra has the structure of an $n! \times n!$ matrix algebra over the symmetric polynomials $P_n^{S_n}$ (cf. Prop. 3.4 of [Rou08]). Among other things, this gives a decomposition of E^n into submodules called 'divided powers':

$$E^n \xrightarrow{\sim} \overbrace{E^{(n)} \oplus \cdots \oplus E^{(n)}}^{n \text{ copies}}.$$

We will make use of this for $n = 2$, where the isomorphism is given (by extension to left $A[y]$-modules) explicitly as follows:

$$E^2[y] \xrightarrow[\left(\begin{smallmatrix} \tau y_1 \\ \tau \end{smallmatrix}\right)]{\sim} \tau y_1 E^2[y] \oplus \tau y_1 E^2[y]. \tag{2.3}$$

The inverse of this map is $(\iota, -y_2)$, where $\iota : \tau y_1 E^2[y] \hookrightarrow E^2[y]$ is the inclusion. The elements τy_1 and $-y_2\tau$ are orthogonal idempotents summing to Id, and τ gives an isomorphism from $-y_2\tau E^2[y]$ to $\tau y_1 E^2[y]$. (To check this: $\tau(-y_2\tau)ee = \tau ee = \tau y_1(\tau ee) \in \tau y_1 E^2[y]$, $\tau(-y_2\tau)y_1 ee = \tau y_1 ee$, and if $\tau(-y_2\tau)ee = 0$ then $\tau ee = 0$ so $-y_2\tau ee = 0$.)

Lemma 2.12 (Lem. 3.12 of [McM22])**.** *There is an isomorphism $R \xrightarrow{\sim} X'_2 \oplus X'_2$ in B-cplx given by the above isomorphism on the degree 0 term of the top row, and the identity on all other terms. So R is a finite direct sum of summands of X. In particular, R is strictly perfect.*

Lemma 2.13 (Lem. 3.13 of [McM22])**.** *There is a quasi-isomorphism $R \xrightarrow{q.i.} E'X_2$ determined by $Id_{E^2[y]}$ on the degree 0 term of the top row and $\left(\begin{smallmatrix} 1 & 0 \\ 1 & -y_1 \end{smallmatrix}\right)$ on the degree 1 term of the bottom row.*

We recall, finally, the main construction from [McM22]. That construction was given using an equivalence:

$$\mathrm{per}\, B \xrightarrow[\mathscr{H}om_B(X,-)]{\sim} \mathrm{per}\, C,$$

where $C = \mathrm{End}_{K^b(B)}(X_1 \oplus X_2)^{\mathsf{op}}$. The algebra C can be presented using a matrix of $(A[y], A[y])$-bimodules (see Prop. 3.31 and Lem. 3.34 of [McM22]):

$$[C] \xrightarrow{\sim} \begin{pmatrix} \mathrm{End}(X_1)^{\mathsf{op}} & \mathrm{Hom}(X_1, X_2) \\ \mathrm{Hom}(X_2, X_1) & \mathrm{End}(X_2)^{\mathsf{op}} \end{pmatrix} \xrightarrow{\sim} \begin{pmatrix} A[y] & y_1 E[y] \\ F[y] & G_1^{\mathsf{op}} \end{pmatrix}.$$

The functor $E' \otimes -$ on per B translates to the functor $\mathscr{E} \otimes -$ on per C, with $\mathscr{E} = \mathscr{H}om_B(X, E'X)$, and there is a quasi-isomorphism $\tilde{E} \xrightarrow{q.i.} \mathscr{E}$ with $\tilde{E} = \mathrm{Hom}_{K^b(B)}(X, E'X)$. This $\tilde{E}$ is a (C, C)-bimodule. It can be presented as a matrix of $(A[y], A[y])$-bimodules (see §3.4.2 of [McM22]):

$$[\tilde{E}] \xrightarrow{\sim} \begin{pmatrix} y_1 E[y] & y_1 y_2 E^2[y] \\ G_1 & G_2 \end{pmatrix}. \tag{2.4}$$

Using the derived equivalence we also have an isomorphism $\tilde{E}^2 = \tilde{E} \otimes_C \tilde{E} \xrightarrow{\sim} \mathrm{Hom}_{K^b(B)}(X, E'^2 X)$, which yields a matrix presentation:

$$[\tilde{E}^2] \xrightarrow{\sim} \begin{pmatrix} y_1 y_2 E^2[y] & y_1 y_2 y_3 E^3[y] \\ G_2 & G_3 \end{pmatrix}. \tag{2.5}$$

Lastly, in §4 of [McM22], the author defined (C, C)-bimodule endomorphisms $\tilde{x}$ and $\tilde{\tau}$. They are given componentwise by:

$$[\tilde{x}] \circlearrowright [\tilde{E}] \quad \text{by:} \quad \begin{pmatrix} \underset{\substack{(\theta,\varphi)\mapsto \\ (y\theta, x\circ\varphi)}}{x} & \underset{\substack{(e_1,e_2,\xi)\mapsto \\ (ye_1, xe_2, xE\circ\xi)}}{xE} \end{pmatrix}, \tag{2.6}$$

$$[\tilde{\tau}] \circlearrowright [\tilde{E}^2] \quad \text{by:} \quad \begin{pmatrix} \underset{\substack{(e_1,e_2,\xi)\mapsto \\ (e',e',\tau\circ\xi)}}{\tau} & \underset{\substack{(ee_1,ee_2,ee_3,\chi)\mapsto \\ (ee',ee',\tau(ee_3),\tau E\circ\chi)}}{\tau E} \end{pmatrix}. \tag{2.7}$$

In the last row, e' is determined by $e_1 - e_2 = y_1 e'$, and ee' is determined by $ee_1 - ee_2 = y_2 ee'$. (See Prop. 2.9 above.) In [McM22] it was established that these endomorphisms satisfy the nil-affine Hecke relations (1.1).

3. More bimodules

We add a new series of bimodules for this paper:

Definition 3.1. Let L_n denote $\mathrm{Hom}_{D^b(B)}(E'^n X_1, X_2)$.

Note that $L_1 = G_1$. We will only need L_1 and L_2 in what follows. Observe that L_n has a right G_1^{op}-module structure given by post-composition. We now study L_2 and provide it with the structure of a $(G_1^{\mathsf{op}}, G_1^{\mathsf{op}})$-bimodule.

We need an additional feature of the complex R:

Lemma 3.2. *The complex R carries a right action of the algebra G_1^{op}, where $(\theta, \varphi) \in G_1^{\mathsf{op}}$ acts by post-composing with $E\varphi \in \mathrm{End}(E^2[y])$ on the top row of R_0, namely $E^2[y]$, and by the matrix*

$$\Phi = \begin{pmatrix} \varphi & 0 \\ \varphi_1 & \theta \end{pmatrix}$$

on the bottom row of R_1, namely $E[y]^{\oplus 2}$, and by $E_y\Phi$ on the top row of R_1, namely $E_y E^{\oplus 2}$. Through the quasi-isomorphism of Lemma 2.13, this action *induces the canonical action of* $G_1^{\mathsf{op}} = \mathrm{End}_{K^b(B)}(X_2)^{\mathsf{op}}$ *on* $E'X_2$ *given by functoriality of* E'.

Proof. First we check that the right action of (θ, φ) described in the lemma gives a morphism of complexes of left B-modules. The action is clearly $A[y]$-linear in the top and bottom rows, and it is clearly linear over the off-diagonal generators in $E_y \subset B$. The action commutes with the differential on the bottom row. We check the top row:

$$\begin{aligned} \begin{pmatrix} E_y\varphi & 0 \\ E_y\varphi_1 & E_y\theta \end{pmatrix} \cdot \begin{pmatrix} \pi E \\ \pi E \circ \tau \end{pmatrix} &= \begin{pmatrix} E_y\varphi \circ \pi E \\ E_y\varphi_1 \circ \pi E + E_y\theta \circ \pi E \circ \tau \end{pmatrix} \\ &= \begin{pmatrix} \pi E \circ E\varphi \\ \pi E \circ E\varphi_1 + \pi E \circ \tau \circ E\theta \end{pmatrix} \\ &= \begin{pmatrix} \pi E \\ \pi E \circ \tau \end{pmatrix} \circ E\Phi. \end{aligned}$$

Next we check that the action commutes with multiplication in the algebra. In G_1^{op} we have $(\theta, \varphi) \cdot (\theta', \varphi') = (\theta\theta', \varphi' \circ \varphi)$, while the action of the product on the bottom row of R_1 is given by:

$$\begin{pmatrix} \varphi' & 0 \\ \varphi_1' & \theta' \end{pmatrix} \cdot \begin{pmatrix} \varphi & 0 \\ \varphi_1 & \theta \end{pmatrix} = \begin{pmatrix} \varphi' \circ \varphi & 0 \\ \varphi_1' \circ \varphi + (\_.\theta') \circ \varphi_1 & \theta\theta' \end{pmatrix}.$$

Note that

$$\varphi' \circ \varphi - \_.\theta\theta' = y_1\big((\_.\theta') \circ \varphi_1 + \varphi_1' \circ \varphi\big),$$

so the composition of the actions agrees with the action of the product on that term. The other terms are trivial to check.

Lastly we check that through the quasi-isomorphism of Lemma 2.13, this action is compatible with the canonical action on $E'X_2$. Start with the bottom row of R_1:

$$\begin{pmatrix} 1 & 0 \\ 1 & -y_1 \end{pmatrix} \cdot \begin{pmatrix} \varphi & 0 \\ \varphi_1 & \theta \end{pmatrix} = \begin{pmatrix} \varphi & 0 \\ \varphi - y_1\varphi_1 & -y_1\theta \end{pmatrix},$$

$$\begin{pmatrix} \varphi & 0 \\ 0 & \theta \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1 & -y_1 \end{pmatrix} = \begin{pmatrix} \varphi & 0 \\ \theta & -\theta y_1 \end{pmatrix}.$$

These agree because $\varphi - y_1\varphi_1 = \theta$. The other terms are trivial to check. □

Now we compute a model for L_2 using the strictly perfect R as a replacement for $E'X_2$.

Definition 3.3. Define the following $(A[y], A[y])$-sub-bimodule of $F[y]^{\oplus 2} \oplus \mathrm{Hom}_A({}_AE^2, E)[y]$:

$$\bar{L}_2 = \Big\langle (f', f, \rho) \in F[y]^{\oplus 2} \oplus \mathrm{Hom}_A({}_AE^2, E)[y] \;\Big|\; \rho = Ef + Ef' \circ \tau + y_1 \circ \rho' \text{ for some } \rho' \in \mathrm{Hom}_A({}_AE^2, E)[y] \Big\rangle.$$

One easily checks that the set $\bar{L}_2$ is closed under the bimodule operations.

Proposition 3.4. *There is an isomorphism of $(A[y], A[y])$-bimodules $\bar{L}_2 \xrightarrow{\sim} \operatorname{Hom}_{K^b(B)}(R, X_2)$ determined by equivariance over $E_y \subset B$ with the following data:*

$$(f', f, \rho) \mapsto \left(\begin{pmatrix} (ee, \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)) \\ (0, \left(\begin{smallmatrix} e \\ e' \end{smallmatrix}\right)) \end{pmatrix} \mapsto \begin{pmatrix} (\rho(ee), 0) \\ (0, f(e) + f'(e')) \end{pmatrix} \right).$$

Proof. The proof is seen by directly computing $Z^0 \mathscr{H}om_B(R, X_2)$. It is easy to check that the morphism given as the image of (f', f, ρ) is a morphism of complexes of left B-modules. The condition $\rho = Ef + Ef' \circ \tau + y_1 \circ \rho'$ is equivalent to the statement that this morphism has zero differential. □

(Recall the notation from [McM22]: ee is an arbitrary element of $E^2[y]$, not a simple tensor. It is unrelated to e and e', which are arbitrary in $E[y]$.)

Corollary 3.5. *The isomorphism above, followed by the canonical isomorphism of functors $\operatorname{Hom}_{K^b(B)}(R, -) \xrightarrow{\sim} \operatorname{Hom}_{D^b(B)}(R, -)$ applied to X_2, gives an isomorphism $\bar{L}_2 \xrightarrow{\sim} L_2$ of $(A[y], A[y])$-bimodules.*

Proposition 3.6. *There is an isomorphism of $(A[y], A[y])$-bimodules $F^2[y] \xrightarrow{\sim} \operatorname{Hom}_{K^b(B)}(R, X_1)$ determined by equivariance over $E_y \subset B$ with the following data:*

$$F^2[y] \ni ff \mapsto \left(\begin{pmatrix} (ee, \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)) \\ (0, \left(\begin{smallmatrix} e \\ e' \end{smallmatrix}\right)) \end{pmatrix} \mapsto \begin{pmatrix} ff(ee) \\ 0 \end{pmatrix} \right).$$

Proof. The proof is seen by directly computing $Z^0 \mathscr{H}om_B(R, X_1)$. □

It is useful to give a model of G_2 that is compatible with this model of L_2 by using the replacement R for $E'X_2$.

Definition 3.7. Define the following $(A[y], A[y])$-sub-bimodule of $E[y]^{\oplus 2} \oplus \operatorname{Hom}_A({}_AE, E^2)[y]$:

$$\begin{aligned} \bar{G}'_2 = \Big\langle (e', e, \xi) \in E[y]^{\oplus 2} \oplus \operatorname{Hom}_A({}_AE, E^2)[y] \Big| \\ \xi = {}_- \otimes e + y_2 \tau ({}_- \otimes (e - y_1 e')) + y_1 y_2 \xi' \\ \text{for some } \xi' \in \operatorname{Hom}_A({}_AE, E^2)[y] \Big\rangle. \end{aligned}$$

One quickly checks that the condition is closed under the bimodule operations. It is sometimes convenient to rewrite the condition as

$$\xi = \tau y_1({}_- \otimes e) - y_2 \tau y_1({}_- \otimes e') + y_1 y_2 \xi'.$$

Proposition 3.8. *There is an isomorphism of $(A[y], A[y])$-bimodules $\bar{G}'_2 \xrightarrow{\sim} \operatorname{Hom}_{K^b(B)}(X_2, R)$ determined by equivariance over $E_y \subset B$ with the following data:*

$$(e', e, \xi) \mapsto \left(\begin{pmatrix} (e, 0) \\ (0, 1) \end{pmatrix} \mapsto \begin{pmatrix} (\xi(e), 0) \\ (0, \left(\begin{smallmatrix} e \\ e' \end{smallmatrix}\right)) \end{pmatrix} \right).$$

Proof. The proof is seen by directly computing $Z^0 \mathscr{H}om_B(X_2, R)$. □

The quasi-isomorphism $R \xrightarrow{q.i.} E'X_2$ determines an isomorphism $\bar{G}'_2 \xrightarrow{\sim} \bar{G}_2$, since X_2 is strictly perfect, given by $(e', e, \xi) \mapsto (e, e - y_1 e', \xi)$, with inverse given by $(e_1, e_2, \xi) \mapsto (y_1^{-1}(e_1 - e_2), e_1, \xi)$. In most of this paper we will use $\bar{G}'_2$ instead of $\bar{G}_2$ as a model for G_2.

Definition 3.9. Let U denote $\mathrm{Hom}_{K^b(B)}(R, R)$. It is canonically isomorphic to $\mathrm{Hom}_{D^b(B)}(E'X_2, E'X_2)$.

Now we describe a model for U. For U and L_2 later in this paper, as for G_n, we frequently assume the terms of the models to denote morphisms of complexes, passing without mention through the isomorphisms $\bar{U} \xrightarrow{\sim} U$ and $\bar{L}_2 \xrightarrow{\sim} L_2$.

Definition 3.10. Define the following $(A[y], A[y])$-sub-bimodule of $FE[y]^{\oplus 4} \oplus \mathrm{Hom}_A({}_AE^2, E^2)[y]$:

$$\bar{U} = \Big\langle (\Phi_{11}, \Phi_{21}, \Phi_{12}, \Phi_{22}, \Lambda) \in FE[y]^{\oplus 4} \oplus \mathrm{Hom}_A({}_AE^2, E^2)[y] \Big| \\ \Lambda = \tau y_1(E\Phi_{11} + E\Phi_{12} \circ \tau) - y_2 \tau y_1 (E\Phi_{21} + E\Phi_{22} \circ \tau) + y_1 y_2 \Lambda^\circ \\ \text{for some } \Lambda^\circ \in \mathrm{Hom}_A({}_AE^2, E^2)[y] \Big\rangle.$$

Here Φ_{ij} give the components of the matrix $[\Phi]$ of a map $\Phi \in \mathrm{End}_A({}_AE[y] \oplus E[y])$. Note that because $y_1 y_2$ is injective, Λ° is uniquely determined by (Φ, Λ). The condition on Λ is clearly closed under the bimodule operations.

Proposition 3.11. *There is an isomorphism of $(A[y], A[y])$-bimodules $\bar{U} \xrightarrow{\sim} U$ determined by equivariance over $E_y \subset B$ with the following data:*

$$(\Phi, \Lambda) \mapsto \left(\begin{pmatrix} (ee, \binom{0}{0}) \\ (0, \binom{e}{e'}) \end{pmatrix} \mapsto \begin{pmatrix} (\Lambda(ee), \binom{0}{0}) \\ (0, [\Phi] \cdot \binom{e}{e'}) \end{pmatrix} \right).$$

Proof. The proof is seen by directly computing $U = Z^0 \mathscr{H}om_B(R, R)$. We must show that the condition on Λ is equivalent to the statement that the image of (Φ, Λ) has zero differential. One computes directly that the morphism given as this image has zero differential if and only if the following pair of equations holds:

$$\begin{cases} \pi E \circ \Lambda = E_y \Phi_{11} \circ \pi E + E_y \Phi_{12} \circ \pi E \circ \tau \\ \pi E \circ \tau \Lambda = E_y \Phi_{21} \circ \pi E + E_y \Phi_{22} \circ \pi E \circ \tau. \end{cases}$$

These are morphisms from $E^2[y]$ in the top row of R_0. On the left side they are given by applying the image of (Φ, Λ) first (namely Λ on $E^2[y]$) and then d. On the right side, $E_y \Phi$ is induced on the top row of R_1 by Φ on the bottom row of R_1 together with equivariance over $E_y \subset B$.

That pair of equations is equivalent to the condition:

$$\begin{cases} \Lambda = E\Phi_{11} + E\Phi_{12} \circ \tau + y_2\Lambda' \\ \tau\Lambda = E\Phi_{21} + E\Phi_{22} \circ \tau + y_2\Lambda'' \end{cases} \tag{3.1}$$
$$\text{for some } \Lambda', \Lambda'' \in \mathrm{Hom}_A({}_AE^2, E^2)[y].$$

For example, the first equation of the pair is equivalent to $\pi E \circ (\Lambda - E\Phi_{11} - E\Phi_{12} \circ \tau) = 0$ because πE commutes with $E_y\Phi_{ij}$. This identity implies the first equation of (3.1) by Lemma 3.7 of [McM22]; cf. also the proof of Prop. 3.26 in that paper.

Claim 3.12. Suppose (Φ, Λ) is given such that (3.1) holds for some Λ', Λ''. Then there is $\Lambda^\circ \in \mathrm{Hom}_A({}_AE^2, E^2)[y]$ such that

$$\Lambda = \tau y_1(E\Phi_{11} + E\Phi_{12} \circ \tau) - y_2\tau y_1(E\Phi_{21} + E\Phi_{22} \circ \tau) + y_1y_2\Lambda^\circ. \tag{3.2}$$

Proof. Multiply the second equation of (3.1) by τ and obtain:

$$-\tau y_2\Lambda'' = \tau \circ E\Phi_{21} + \tau \circ E\Phi_{22} \circ \tau.$$

Multiply the first by τ and the second by τy_1 and identify the results to obtain:

$$\tau y_2\Lambda' = y_1y_2\tau\Lambda'' + \tau y_1 \circ \big(E\Phi_{21} + E\Phi_{22} \circ \tau\big) - \tau \circ \big(E\Phi_{11} + E\Phi_{12} \circ \tau\big).$$

Then:

$$\begin{aligned} \Lambda' &= (y_1\tau - \tau y_2) \circ \Lambda' \\ &= y_1\tau\Lambda' - y_1y_2\tau\Lambda'' - \tau y_1 \circ \big(E\Phi_{21} + E\Phi_{22} \circ \tau\big) + \tau \circ \big(E\Phi_{11} + E\Phi_{12} \circ \tau\big) \\ &= y_1\big(\tau\Lambda' - y_2\tau\Lambda''\big) - \tau y_1 \circ \big(E\Phi_{21} + E\Phi_{22} \circ \tau\big) + \tau \circ \big(E\Phi_{11} + E\Phi_{12} \circ \tau\big). \end{aligned}$$

Let $\Lambda^\circ = \tau\Lambda' - y_2\tau\Lambda''$. Then:

$$\begin{aligned} \Lambda &= E\Phi_{11} + E\Phi_{12} \circ \tau + y_1y_2\Lambda^\circ \\ &\qquad - y_2\tau y_1 \circ \big(E\Phi_{21} + E\Phi_{22} \circ \tau\big) + y_2\tau \circ \big(E\Phi_{11} + E\Phi_{12} \circ \tau\big) \\ &= \tau y_1 \circ (E\Phi_{11} + E\Phi_{12} \circ \tau) - y_2\tau y_1 \circ (E\Phi_{21} + E\Phi_{22} \circ \tau) + y_1y_2\Lambda^\circ, \end{aligned}$$

as desired. □

Claim 3.13. Now suppose (Φ, Λ) and Λ° are given such that (3.2) holds. Then there are Λ', Λ'' such that (3.1) holds.

Proof. Let

$$\begin{aligned} \Lambda' &= \tau \circ \big(E\Phi_{11} + E\Phi_{12} \circ \tau\big) - \tau y_1 \circ \big(E\Phi_{21} + E\Phi_{22} \circ \tau\big) + y_1\Lambda^\circ, \\ \Lambda'' &= \tau \circ \big(E\Phi_{21} + E\Phi_{22} \circ \tau\big) + y_1\tau\Lambda^\circ. \end{aligned}$$

Multiplying the first by y_2, adding $E\Phi_{11} + E\Phi_{12} \circ \tau$, and simplifying with (3.2), we find:

$$y_2\Lambda' + E\Phi_{11} + E\Phi_{12} \circ \tau = \Lambda.$$

Multiplying the second by y_2 and adding $E\Phi_{21} + E\Phi_{22} \circ \tau$, we find:

$$y_2\Lambda'' + E\Phi_{21} + E\Phi_{22} \circ \tau = \tau y_1 \circ \big(E\Phi_{21} + E\Phi_{22} \circ \tau\big) + \tau y_1y_2\Lambda^\circ,$$

while

$$\begin{aligned}\tau\Lambda &= -\tau y_2 \tau y_1 \circ \big(E\Phi_{21} + E\Phi_{22} \circ \tau\big) + \tau y_1 y_2 \Lambda^\circ \\ &= \tau y_1 \circ \big(E\Phi_{21} + E\Phi_{22} \circ \tau\big) + y_1 y_2 \tau \Lambda^\circ\end{aligned}$$

using (3.2). So the pair of equations (3.1) is satisfied. □

The proposition follows. □

We will need one more description of U:

Lemma 3.14. *The composition map $L_2 \otimes_{G_1^{\mathsf{op}}} G_2 \to U$ is an isomorphism.*

Proof. Consider the triangulated functor:

$$\mathscr{H}om_B(X_2, -) : K^b(B) \to K^b(G_1^{\mathsf{op}}).$$

By the same reasoning as in §3.3.2 of [McM22], this functor descends to the derived categories

$$\mathscr{H}om_B(X_2, -) : D^b(B) \to D^b(G_1^{\mathsf{op}}),$$

it is fully faithful when restricted to $\langle X_2 \rangle_\Delta$, and it is essentially surjective from $\langle X_2 \rangle_\Delta$ (because the image of X_2 is quasi-isomorphic to G_1^{op}). The inverse is given by $X_2 \otimes_{G_1^{\mathsf{op}}} -$. It follows from $R \in \langle X_2 \rangle_\Delta$ (Lemma 3.12 of [McM22]) and

$$\begin{aligned}\mathrm{Hom}_{K^b(B)}(X_2, R) &\xrightarrow{\sim} \mathrm{Hom}_{K^b(B)}(X_2, E'X_2) \\ &\xrightarrow{q.i.} \mathscr{H}om_B(X_2, E'X_2) \\ &\xrightarrow{q.i.} \mathscr{H}om_B(X_2, R)\end{aligned}$$

that the evaluation map is an isomorphism:

$$X_2 \otimes_{G_1^{\mathsf{op}}} \mathrm{Hom}_{K^b(B)}(X_2, R) \xrightarrow{\sim} R.$$

This shows that the map in the lemma statement is an isomorphism:

$$\begin{aligned}&\mathrm{Hom}_{K^b(B)}(R, X_2) \otimes_{G_1^{\mathsf{op}}} \mathrm{Hom}_{K^b(B)}(X_2, R) \\ &\xrightarrow{\sim} \mathrm{Hom}_{K^b(B)}\big(R, X_2 \otimes_{G_1^{\mathsf{op}}} \mathrm{Hom}_{K^b(B)}(X_2, R)\big) \\ &\xrightarrow{\sim} \mathrm{Hom}_{K^b(B)}(R, R).\end{aligned}$$

□

We will need to know the $(A[y], A[y])$-bimodule structure of the components of $\tilde{E}$ and $\tilde{E}^2$ and $\tilde{F}$. These may be read off of presentations we have given by using the fact that $y_i = x_i - y$ is injective as an endomorphism of $E^n[y]$ (for any n). We write y_i^{-1} for the inverse morphism defined on the image $y_i E^n[y]$.

Proposition 3.15. *We have isomorphisms of $(A[y], A[y])$-bimodules:*

- $y_1 \dots y_n E^n[y] \xrightarrow{\sim} E^n[y]$ *given by application of* $(y_1 \dots y_n)^{-1}$.
- $L_1 = G_1 \xrightarrow{\sim} A[y] \oplus FE[y]$ *given by* $(\theta, \varphi) \mapsto (\theta, \varphi_1)$*, where*

$$\varphi_1 = y_1^{-1}(\varphi - \theta)$$

is interpreted in $FE[y]$. Note that the summand $FE^2[y]$ is not only a left $A[y]$-submodule of G_2, but moreover a left G_1^{op}-submodule of G_2.

- $G_2 \xrightarrow{\sim} E[y] \oplus E[y] \oplus FE^2[y]$ *given by* $(e', e, \xi) \mapsto (e', e, \xi')$*, where*
$$\xi' = (y_1y_2)^{-1}\big(\xi - {}_{-}\otimes e - y_2\tau({}_{-}\otimes(e - y_1e'))\big)$$
is interpreted in $FE^2[y]$*. Note that the summand* $FE^2[y]$ *is a left* G_1^{op}*-submodule of* G_2*.*
- $L_2 \xrightarrow{\sim} F[y] \oplus F[y] \oplus F^2E[y]$ *given by* $(f', f, \rho) \mapsto (f', f, \rho_1)$*, where*
$$\rho_1 = y_1^{-1}\big(\rho - Ef - Ef' \circ \tau\big)$$
is interpreted in $F^2E[y]$*. Note that the summand* $F^2E[y]$ *is a left* G_1^{op}*-submodule of* L_2*.*
- $U \xrightarrow{\sim} FE[y]^{\oplus 4} \oplus F^2E^2[y]$ *given by*
$$(\Phi_{11}, \Phi_{21}, \Phi_{12}, \Phi_{22}, \Lambda) \mapsto (\Phi_{11}, \Phi_{21}, \Phi_{12}, \Phi_{22}, \Lambda^\circ),$$
where
$$\Lambda = \tau y_1(E\Phi_{11} + E\Phi_{12} \circ \tau) - y_2\tau y_1(E\Phi_{21} + E\Phi_{22} \circ \tau) + y_1y_2\Lambda^\circ \tag{3.3}$$
determines Λ°*, which is interpreted in* $F^2E^2[y]$*. Note that the summand* $F^2E^2[y]$ *is a left* G_1^{op}*-submodule of* U*.*

Proof. The first point is obvious. The second point follows from Prop. 2.9 because (θ, φ_1) may be chosen arbitrarily in $A^{\mathsf{op}}[y] \oplus \mathrm{End}_A({}_AE)[y]$, determining φ; while a given choice of (θ, φ) satisfying the condition determines φ_1 by way of y_1^{-1}. Similar reasoning applies to G_2, L_2, and U, working with Defs. 3.7, 3.3, and 3.10, respectively. □

In what follows, we will frequently use the bimodule descriptions on the right sides of the isomorphisms in Prop. 3.15. Sometimes, to avoid confusion, we will use the shorthand expression 'submodule form' to refer to the left sides of the isomorphisms (i.e. presentations as submodules cut out by conditions, as in the definitions of these structures), and 'bimodule form' to refer to the right sides of the isomorphisms. Considering the component data of an element in one of these structures, the components in submodule form and bimodule form differ only in the last component: in submodule form the last component gives the full morphism on the degree 0 part of the top row of the complex, and in bimodule form the last component gives the remainder term 'φ_1', 'ξ'', 'χ''', 'ρ_1', or 'Λ°' that is produced from the conditions by inverting some y_i.

4. Adjunction

Definition 4.1. Let $\tilde{F}$ denote the (C, C)-bimodule ${}^\vee\tilde{E}$, that is, $\mathrm{Hom}_C({}_C\tilde{E}, C)$.

We know that ${}_C\tilde{E}$ is f.g. projective. It follows that the right adjoint functor $\mathrm{Hom}_C({}_C\tilde{E}, -)$ of $\tilde{E} \otimes_C -$ is canonically isomorphic to $\tilde{F} \otimes_C -$. We have already defined $\tilde{x}$ and $\tilde{\tau}$. We define $\tilde{\varepsilon} : \tilde{E}\tilde{F} \to C$ and $\tilde{\eta} : C \to \tilde{F}\tilde{E}$ using the duality, and then $\tilde{\sigma}$ and $\tilde{\rho}_\lambda$ using the formulas in §2.2 with $(A, E, F, x, \tau, \eta, \varepsilon)$ replaced by $(C, \tilde{E}, \tilde{F}, \tilde{x}, \tilde{\tau}, \tilde{\eta}, \tilde{\varepsilon})$. Note that sometimes we view $\tilde{F}\tilde{E}$ through the canonical isomorphism $\mathrm{Hom}(\tilde{E}, C) \otimes_C \tilde{E} \xrightarrow{\sim} \mathrm{Hom}(\tilde{E}, \tilde{E})$.

Now we construct an isomorphism of (C, C)-bimodules
$$\tilde{F} \xrightarrow{\sim} \mathrm{Hom}_{K^b(B)}(X_2 \oplus R, X)$$

as follows:

$$\begin{aligned}\tilde{F} = &\ \mathrm{Hom}_C({}_C\tilde{E}, C)\\ &\xrightarrow{\sim} \mathrm{Hom}_{D^b(C)}(\tilde{E}, C)\\ &\xrightarrow{\sim}\ \mathrm{Hom}_{D^b(C)}\left(\mathscr{H}om_B(X, E'X), \mathscr{H}om_B(X, X)\right)\\ &\xrightarrow{\sim} \mathrm{Hom}_{D^b(B)}(E'X, X)\\ &\xrightarrow{\sim} \mathrm{Hom}_{D^b(B)}(X_2 \oplus R, X_1 \oplus X_2)\\ &\xrightarrow{\sim} \mathrm{Hom}_{K^b(B)}(X_2 \oplus R, X_1 \oplus X_2).\end{aligned}$$

The second arrow comes from the quasi-isomorphisms of Lemma 3.33 and Corollary 3.41 of [McM22]. The third arrow comes from the equivalence per $B \xrightarrow{\sim}$ per C. (By a similar calculation we have $\tilde{F}^n \xrightarrow{\sim} \mathrm{Hom}_{D^b(B)}(E'^n X, X)$. This explains the use of the derived category for G_n and L_n.) The fourth arrow holds because $R \xrightarrow{q.i.} E'X_2$ (Lemma 2.13), and the fifth holds because R is strictly perfect (Lemma 2.12).

With this description of $\tilde{F}$, we can express it as a 2×2 matrix of $(A[y], A[y])$-bimodules whose entries we have studied:

$$[\tilde{F}] \xrightarrow{\sim} \begin{pmatrix}\mathrm{Hom}(X_2, X_1) & \mathrm{Hom}(X_2, X_2)\\ \mathrm{Hom}(R, X_1) & \mathrm{Hom}(R, X_2)\end{pmatrix} \xrightarrow{\sim} \begin{pmatrix}F[y] & L_1\\ F^2[y] & L_2\end{pmatrix}. \tag{4.1}$$

The top row of $[\tilde{F}]$ has been computed as the bottom row of $[C]$. We found $\mathrm{Hom}_{K^b(B)}(R, X_1)$ in Prop. 3.6, and we found $\mathrm{Hom}_{K^b(B)}(R, X_2)$ in Prop. 3.4.

We have $C = \mathrm{End}_{K^b(B)}(X_1 \oplus X_2)$, and the right action of C on $\tilde{F}$ is given by post-composition. The left action of C is by pre-composition, but one must first apply functoriality of E' and use the quasi-isomorphism from Lemma 2.13, which we write $\gamma : R \xrightarrow{q.i.} E'X_2$; we have:

- A generator $\phi \in Z^0\mathscr{H}om_B(X_1, X_1)^{\mathsf{op}} \cong A[y] \subset C$ determines $E'\phi \in \mathrm{Hom}_{K^b(B)}(X_2, X_2)$ that acts on $\tilde{F}$ (on the top row) by pre-composition. An element $\phi = \theta \in A[y]$ acts in the obvious way on the left on $F[y]$ and L_1.
- A generator $\phi \in Z^0\mathscr{H}om_B(X_2, X_1) \cong F[y] \subset C$ determines

  $$E'\phi \in \mathrm{Hom}_{D^b(B)}(E'X_2, E'X_1) \xrightarrow[\sim]{-\circ\gamma} \mathrm{Hom}_{K^b(B)}(R, X_2).$$

  So ϕ acts on $\tilde{F}$ by pre-composition with $E'\phi\circ\gamma : R \to X_2$, taking the top row to the bottom row. Recall that we have the model $\bar{L}_2$ for $\mathrm{Hom}_{K^b(B)}(R, X_2)$. An element $\phi = f \in F[y]$ acts by pre-composition with the morphism determined by $E'\phi \circ \gamma = (0, f, 0) \in \bar{L}_2$.
- A generator $\phi \in Z^0\mathscr{H}om_B(X_1, X_2) \cong y_1E[y] \subset C$ determines

  $$E'\phi \in \mathrm{Hom}_{K^b(B)}(E'X_1, E'X_2) \xleftarrow[\sim]{\gamma\circ -} \mathrm{Hom}_{K^b(B)}(X_2, R).$$

  Recall that we have the models $\bar{G}_2$ for $\mathrm{Hom}_{K^b(B)}(X_2, E'X_2)$ and $\bar{G}_2'$ for $\mathrm{Hom}_{K^b(B)}(X_2, R)$, and the isomorphism $\bar{G}_2 \xrightarrow{\sim} \bar{G}_2'$ given by $(e_1, e_2, \xi') \mapsto (y_1^{-1}(e_1 - e_2), e_1, \xi')$ (in bimodule forms). An element $\phi = y_1e \in y_1E[y]$ determines $E'\phi = (y_1e, 0, 0) \in \bar{G}_2$, so this acts on $\tilde{F}$ by pre-composition with the morphism determined by $(e, y_1 e, 0) \in \bar{G}'_2$, taking the bottom row to the top row.

- A generator
$$\phi \in Z^0 \mathscr{H}om_B(X_2, X_2)^{\mathsf{op}} \cong G_1^{\mathsf{op}} \subset C$$
determines $\phi_R \in \mathrm{Hom}_{K^b(B)}(R, R)$ from the right action of G_1^{op} on R. In terms of the model $\bar{U}$, we have $\phi_R = (\varphi, \varphi_1, 0, \theta, E\varphi)$ (in submodule form), determined by $\phi = (\theta, \varphi) \in G_1^{\mathsf{op}}$. This acts on $\tilde{F}$ (on the bottom row) by pre-composition.

5. Isomorphisms $\tilde{\rho}_\lambda$

5.1. **Some tensor products of (C, C)-bimodules.** In this section we compute three tensor products of bimodules over C, namely $\tilde{E}\tilde{E}$, $\tilde{F}\tilde{E}$, and $\tilde{E}\tilde{F}$, and describe the products in each case as matrices of $(A[y], A[y])$-bimodules. These calculations are used in the remaining sections to verify that $\tilde{\rho}_\lambda$ are isomorphisms. Note that the product $\tilde{E}\tilde{E} = \tilde{E}^2$ is already given a description (Eq. 2.5) as a matrix of $(A[y], A[y])$-bimodules using the identification with $\mathrm{Hom}_{K^b(B)}(X, E'^2 X)$, but in order to compute $\tilde{\sigma}$ it is also necessary to realize $\tilde{E}^2$ as the tensor product over C of the bimodule $\tilde{E}$ with itself.

These tensor products are computed according to the general formulation described in §2.4 of [McM22]. First we take the tensor product over the subalgebra $\Delta := \begin{pmatrix} A[y] & 0 \\ 0 & G_1^{\mathsf{op}} \end{pmatrix} \subset C$. This product is given on components by matrix multiplication and tensor product over $A[y]$ or G_1^{op}. After this we must take the quotient by $\mathrm{Im}(I_{y_1 E[y]}) + \mathrm{Im}(I_{F[y]})$, where $I_{y_1 E[y]}$ and $I_{F[y]}$ apply the actions of the off-diagonal generators in C. This quotient may be taken separately on each coefficient of the product over Δ.

Using the language of §2.4 of [McM22] with some given $M_R = \begin{pmatrix} M_1 & M_2 \end{pmatrix}$, ${}_R N = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}$, and $R = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, the simplest technique at our disposal for computing a quotient by the image of (say) I_B is to identify one of its projections as an isomorphism. (In §2.4 of [McM22], there is, for example, a projection of I_B to $M_1 \otimes_A N_1$ and another projection to $M_2 \otimes_D N_2$.) In this situation the quotient by $\mathrm{Im}(I_B)$ reduces to the summand of the second projection, because every element of the first summand (in the quotient) has a unique representative in the second summand. If it also happens that $\mathrm{Im}(I_C) \subset \mathrm{Im}(I_B)$, then the quotient by the sum $\mathrm{Im}(I_B) + \mathrm{Im}(I_C)$ is still isomorphic to the second summand. Many of the components computed below are found in this way, but a few of them require more complicated reasoning.

Let us write, in general, I'_β for the projection of I_B to the first summand, and $-I''_\beta$ for the projection to the second. Similarly write I'_δ and $-I''_\delta$ for the projections of I_C. Here 'first' and 'second' summand and 'I_B' and 'I_C' are understood as in §2.4 of [McM22]. In a tensor product of (C, C)-bimodules, each of the four coefficients will have its own set of maps I'_β, I''_β, I'_δ, I''_δ.

The matrix forms of the bimodules $\tilde{E}$, $\tilde{E}^2$, and $\tilde{F}$ are given in Eqs. (2.4), (2.5), and (4.1). For some of our calculations it helps to be clear about the structures of the component bimodules, so we translate the components to the bimodule forms on the right sides of the isomorphisms in Prop. 3.15. Note the consequence that the formulas for multiplication within C and for the actions of elements of C on components of $\tilde{E}$ or $\tilde{F}$ are more complicated; this is illustrated, for example, in the formula for $\Gamma_{12}\,|_{E^2[y]_G G_2}$ in the next section.

5.1.1. $\tilde{E}\tilde{E}$. For the product $\tilde{E}\tilde{E}$, we already know the structures of the coefficients of the matrix presentation from Eq. (2.5) (and Prop. (3.15)). We will need to compute the action of $\tilde{\tau}$ on elements of $\tilde{E}\tilde{E}$ in order to compute $\tilde{\sigma}$, and for this it will be necessary and sufficient to identify the map from the tensor product over Δ to the tensor product over C, i.e. to the quotient by $\operatorname{Im}(I'_\beta - I''_\beta) + \operatorname{Im}(I'_\delta - I''_\delta)$. Write Γ for this map. Let the subscript 'G' between concatenated modules indicate the tensor product over G_1^{op}. (An empty subscript indicates the product over $A[y]$.) So we have:

$$\begin{aligned} \tilde{E}\otimes_\Delta \tilde{E} &\cong \begin{pmatrix} E[y] & E^2[y] \\ G_1 & G_2 \end{pmatrix} \otimes_\Delta \begin{pmatrix} E[y] & E^2[y] \\ G_1 & G_2 \end{pmatrix} \\ &\cong \begin{pmatrix} EE[y]\oplus E^2[y]_G G_1 & EE^2[y]\oplus E^2[y]_G G_2 \\ G_1E[y]\oplus (G_2)_G G_1 & G_1E^2[y]\oplus (G_2)_G G_2 \end{pmatrix} \\ &\xrightarrow{\Gamma} \begin{pmatrix} E^2[y] & E^3[y] \\ G_2 & G_3 \end{pmatrix} \cong [\tilde{E}^2], \end{aligned} \tag{5.1}$$

and we wish to understand the map Γ on each component. We find these component maps for simple tensors using the following steps. First we interpret a pair of elements of the left and right tensor factors $\tilde{E}$ as morphisms in $Z^0\mathscr{H}om_B(X, E'X)$ using isomorphisms such as Prop. 3.18 in [McM22]. Then we apply E' to the morphism of the right factor, and post-compose the result with the morphism of the left, obtaining an element of $Z^0\mathscr{H}om_B(X, E'^2X)$. That element is interpreted again in $\tilde{E}^2$. (See Prop. 3.37 and Lemmas 3.44 and 3.45 of [McM22].)

To facilitate checking these steps, the reader is encouraged to write out the complexes for X, $E'X$, and E'^2X, and to be familiar with Prop. 3.15 and the interpretations of elements of the structures in that proposition as homomorphisms of complexes. With this in mind, the calculations are mechanical, if tedious. We demonstrate the first cases with detailed explanation, and for the remaining cases we record the results.

- For Γ_{11}, we have:
  - $\Gamma_{11}\,|_{EE[y]}$ is given by $\mathrm{Id}_{EE[y]}$.

    To see this, let e_1 represent an element of the left factor $E[y]$, and e_2 an element of the right factor $E[y]$. (We are suppressing the isomorphism $EE[y] \xrightarrow{\sim} E[y]E[y]$.) Viewed in $[\tilde{E}]_{11}$ through Prop. 3.15, these correspond to y_1e_1 and y_1e_2. As a homomorphism of complexes, y_1e_2 sends $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \in X_1$ to $\left(\begin{smallmatrix}(y_1e_2,0)\\(0,0)\end{smallmatrix}\right) \in E'X_1 = X_2$. (Use Prop. 3.31, Lemma 3.34, and Lemma 3.7 of [McM22].) Applying functoriality (with Lemma 3.8 of [McM22] in mind to notate $E'X_2$), we have $E'(y_1e_2) : X_2 \to E'X_2$ by a map that (among other things) takes the element $\begin{pmatrix} (y_1e_1,0) \\ (0,0) \end{pmatrix}$ to

$$\begin{pmatrix} (y_1e_1 \otimes y_1e_2, \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right), 0) \\ (0, \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right), 0) \end{pmatrix} = \begin{pmatrix} (y_1y_2(e_1 \otimes e_2), \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right), 0) \\ (0, \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right), 0) \end{pmatrix},$$

which is therefore the image of $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \in X_1$ under $E'(y_1e_2) \circ y_1e_1 : X_1 \to E'X_2$. By Lemma 3.47 of [McM22] combined with Prop. 3.15, this image corresponds to $e_1 \otimes e_2$ in $E^2[y] \cong [\tilde{E}^2]_{11}$, so Γ_{11} is the identity.

– $\Gamma_{11} \mid_{E^2[y]_G G_1}$ is given as the inverse of $E^2[y] \xrightarrow{\sim} E^2[y]_G G_1$, $ee \mapsto ee \otimes 1_{G_1}$.

Here ee corresponds to a map $X_1 \to E'X_2$. The right action of $g \in G_1$ on $E^2[y]$ is given in terms of maps of complexes by post-composing with the induced map $E'(g) : E'X_2 \to E'X_2$, but this is also the effect of Γ_{11} on terms in $E^2[y]_G G_1$. (The apparent coincidence derives from the definition $X_2 = E'X_1$ and the matrix descriptions of C and $\tilde{E}$.)

- For Γ_{21}, we have:
  – $\Gamma_{21} \mid_{G_1 E[y]}$ is given (in bimodule forms) by

$$(\theta, \varphi_1) \otimes e \mapsto (\theta e, \theta y_1 e, \varphi_1(-) \otimes e) \in G_2.$$

The map $(\theta, \varphi_1) : X_2 \to X_2$ is determined (as in Prop. 3.18 of [McM22] except in bimodule form) by:

$$\begin{pmatrix} (e_1, 0) \\ (0, 1) \end{pmatrix} \overset{(\theta,\varphi_1)}{\longmapsto} \begin{pmatrix} (e_1.\theta + y_1\varphi_1(e_1), 0) \\ (0, \theta) \end{pmatrix}.$$

Further, e corresponds to the map $e : X_1 \to X_2$ given by $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \mapsto \begin{pmatrix} (y_1e,0) \\ (0,0) \end{pmatrix}$, which by functoriality induces a map $E'(e) : E'X_1 \to E'X_2$ that is given (similarly to $\Gamma_{11} \mid_{EE[y]}$ above) by:

$$\begin{pmatrix} (e_1, 0) \\ (0, 1) \end{pmatrix} \overset{E'(e)}{\longmapsto} \begin{pmatrix} (e_1 \otimes y_1 e, \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right), 0) \\ (0, \left(\begin{smallmatrix}y_1e\\0\end{smallmatrix}\right), 0) \end{pmatrix}.$$

Therefore the composition is given by:

$$\begin{pmatrix} (e_1, 0) \\ (0, 1) \end{pmatrix} \overset{E'(e)\circ(\theta,\varphi_1)}{\longmapsto} \begin{pmatrix} (e_1.\theta \otimes y_1 e + y_1\varphi_1(e_1) \otimes y_1 e, \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right), 0) \\ (0, \left(\begin{smallmatrix}\theta y_1e\\0\end{smallmatrix}\right), 0) \end{pmatrix}.$$

This image is in $E'X_2$. The map corresponds to the element

$$(\theta y_1 e, 0, \_ \otimes \theta y_1 e + y_1y_2(\varphi_1(-) \otimes e) \in \bar{G}_2$$

(written in submodule form), which translates (see the paragraph after Prop. 3.8) to the element

$$(\theta e, \theta y_1 e, \varphi_1(-) \otimes e) \in \bar{G}_2'$$

(written in bimodule form) considering Def. 3.7 and Prop. 3.8 (and Prop. 3.15 for the bimodule form); this is the formula we wished to establish.

– $\Gamma_{21}|_{(G_2)_G G_1}$ is given as the inverse of $G_2 \xrightarrow{\sim} (G_2)_G G_1$, $g_2 \mapsto g_2 \otimes 1_{G_1}$.

This is similar to $\Gamma_{11}|_{E^2[y]_G G_1}$ above.

- For Γ_{12}, we have:
  – $\Gamma_{12}|_{EE^2[y]}$ is given by $\mathrm{Id}_{E^3[y]}$.

  We leave this and the remaining calculations to the reader, and record the results.

  – $\Gamma_{12}|_{E^2[y]_G G_2}$ is given (in bimodule forms) by
$$ee \otimes (e', e, \xi') \mapsto (y_1y_2y_3)^{-1}(E\xi)(y_1y_2ee) \in E^3[y].$$
- For Γ_{22}, we have:
  – $\Gamma_{22}|_{G_1E^2[y]}$ is given by
$$(\theta, \varphi_1) \otimes ee \mapsto (\theta y_1y_2ee, 0, 0, \varphi_1(-) \otimes ee) \in G_3.$$

  A remark: in §4.2 of [McM22], the horizontal arrows of the diagrams are based on the same type of calculations, except that submodule forms are used. For example, the map specified just above is $E'(ee) \circ (\theta, \varphi_1)$. In submodule form it would be written $E'(y_1y_2ee) \circ (\theta, \varphi)$ with $\varphi = \_.\theta + y_1\varphi_1$, and this results from the horizontal arrow in Diagram $D_{1|2}(2, 1, 1)$. So this composite would be written in G_3 in submodule form as
$$E'(y_1y_2ee) \circ (\theta, \varphi) = (\theta y_1y_2ee, 0, 0, \varphi \otimes ee).$$

  We encourage the reader to check §4.2 of [McM22] for occasional further hints regarding some of these calculations.

  – $\Gamma_{22}|_{(G_2)_G G_2}$ is given (in submodule forms) by
$$(e_1, e_2, \xi) \otimes (\bar{e}_1, \bar{e}_2, \bar{\xi}) \mapsto (\bar{\xi}(e_1), e_2 \otimes \bar{e}_1, e_2 \otimes \bar{e}_2, E\bar{\xi} \circ \xi) \in G_3$$
  (c.f. Diagram $D_{1|2}(2, 2, 1)$). We will need to have this map written for the bimodule forms. First translate the notation from $\bar{G}_2$ to $\bar{G}_2'$ (cf. the paragraph after Prop. 3.8) using e', e for the first factor (so $e' = y_1^{-1}(e_1 - e_2)$ and $e = e_1$) and $\bar{e}'$, $\bar{e}$ for the second factor. Then expand ξ in terms of e', e, and ξ' according to the condition for elements of G_2 (Def. 3.7), and likewise expand $\bar{\xi}$. Now compute $E\bar{\xi} \circ \xi$:
$$\begin{aligned}
E\bar{\xi} \circ \xi = &\big(_\,_ \otimes \bar{e} + y_2\tau(_\,_ \otimes (\bar{e} - y_1\bar{e}')) + y_1y_2E\bar{\xi}'\big) \\
&\circ \big(_ \otimes e + y_2\tau(_ \otimes (e - y_1e')) + y_1y_2\xi'\big) \\
&= _ \otimes \big(e\bar{e} + y_2\tau(e\bar{e} - y_1e\bar{e}') + y_1y_2\bar{\xi}'(e)\big) \\
&+ y_3 \circ \tau E\big(_ \otimes (e\bar{e} - y_2e'\bar{e})\big) \\
&+ y_2y_3 \circ E\tau \circ \tau E\big(_ \otimes (e - y_1e')(\bar{e} - y_1\bar{e}')\big) \\
&+ y_1y_2y_3\big(E\bar{\xi}' \circ \tau(_ \otimes (e - y_1e')) + E\tau(\xi' \otimes (\bar{e} - y_1\bar{e}')) + \xi' \otimes \bar{e}'\big).
\end{aligned}$$

To find χ'', subtract all but the last term of χ in the condition of $\bar{G}_3$ in Prop. 2.9 and remove $y_3y_2y_1$. Obtain the complete image in bimodule form:

$$\begin{aligned}&(e', e, \xi') \otimes (\bar{e}', \bar{e}, \bar{\xi}') \mapsto \\ &\big(e\bar{e} + y_2\tau(e\bar{e} - y_1 e\bar{e}') + y_1y_2\bar{\xi}'(e), e\bar{e} - y_2e'\bar{e}, (e - y_1e')(\bar{e} - y_1\bar{e}'), \\ &\quad E\bar{\xi}' \circ \tau(\_ \otimes (e - y_1e')) + E\tau(\xi' \otimes (\bar{e} - y_1\bar{e}')) + \xi' \otimes \bar{e}'\big).\end{aligned}$$

5.1.2. $\tilde{F}\tilde{E}$. For the product $\tilde{F}\tilde{E}$, we can find the $(A[y], A[y])$-bimodule structure of the components of its matrix presentation using the same technique as for $\tilde{F}$ and $\tilde{E}^2$. We have:

$$\begin{aligned}\tilde{F}\tilde{E} &= \operatorname{Hom}_C({}_C\tilde{E}, C) \otimes_C \tilde{E} \\ &\xrightarrow{\sim} \operatorname{Hom}_C({}_C\tilde{E}, \tilde{E}) \xrightarrow{\sim} \operatorname{Hom}_{D^b(C)}(\tilde{E}, \tilde{E}) \xrightarrow{\sim} \operatorname{Hom}_{D^b(C)}(\mathscr{E}, \mathscr{E}) \\ &\xrightarrow{\sim} \operatorname{Hom}_{D^b(C)}\left(\mathscr{H}om_B(X, E'X), \mathscr{H}om_B(X, E'X)\right) \\ &= \operatorname{Hom}_{D^b(B)}(E'X, E'X) \\ &\xrightarrow{\sim} \operatorname{Hom}_{K^b(B)}(X_2 \oplus R, X_2 \oplus R).\end{aligned}$$

(The last isomorphism uses the quasi-isomorphism $R \overset{q.i.}{\to} E'X_2$ and the fact that $E'X_1 = X_2$ and R are strictly perfect.) So the matrix presentation is:

$$[\tilde{F}\tilde{E}] \xrightarrow{\sim} \begin{pmatrix} \operatorname{Hom}(X_2, X_2) & \operatorname{Hom}(X_2, R) \\ \operatorname{Hom}(R, X_2) & \operatorname{Hom}(R, R) \end{pmatrix} \xrightarrow{\sim} \begin{pmatrix} G_1 & G_2 \\ L_2 & U \end{pmatrix}. \tag{5.2}$$

As we did for $\tilde{E}^2$, we study the map Γ from the components of the product over Δ to those of the product over C:

$$\begin{aligned}\tilde{F} \otimes_\Delta \tilde{E} &\cong \begin{pmatrix} F[y] & L_1 \\ F^2[y] & L_2 \end{pmatrix} \otimes_\Delta \begin{pmatrix} E[y] & E^2[y] \\ G_1 & G_2 \end{pmatrix} \\ &\cong \begin{pmatrix} FE[y] \oplus (L_1)_G G_1 & FE^2[y] \oplus (L_1)_G G_2 \\ F^2E[y] \oplus (L_2)_G G_1 & F^2E^2[y] \oplus (L_2)_G G_2 \end{pmatrix} \\ &\cong \begin{pmatrix} FE[y] \oplus G_1 & FE^2[y] \oplus G_2 \\ F^2E[y] \oplus L_2 & F^2E^2[y] \oplus (L_2)_G G_2 \end{pmatrix} \xrightarrow{\Gamma} \begin{pmatrix} G_1 & G_2 \\ L_2 & U \end{pmatrix}.\end{aligned} \tag{5.3}$$

The bulleted claims below are justified in the paragraphs following them.

- We have $\Gamma_{11} : FE[y] \oplus G_1 \to G_1$ given by $(\iota, \operatorname{Id}_{G_1})$.

  Here the map $\iota : FE[y] \hookrightarrow L_1 = G_1$ is the inclusion of the second summand as written in Prop. 3.15.

  - $I'_\beta : FE[y]_G G_1 \xrightarrow{\sim} FE[y]$ given as the inverse of the isomorphism $\big(fe \mapsto fe \otimes 1_{G_1}\big)$,
  - $I''_\beta : FE[y]_G G_1 \xrightarrow{\iota \otimes G_1} (L_1)_G G_1 \cong G_1$,
  - $I'_\delta : (G_1)_G FE[y] \xrightarrow{\sim} FE[y]$ given as the inverse of the isomorphism $\big(fe \mapsto 1_{G_1} \otimes fe\big)$,
  - $I''_\delta : (G_1)_G FE[y] \xrightarrow{G_1 \otimes \iota} (G_1)_G L_1 \cong G_1$.

Using either $I''_\beta \circ I'^{-1}_\beta$, or $I''_\delta \circ I'^{-1}_\delta$, one associates a unique representative in $(L_1)_G G_1 \cong G_1$ to each element of $FE[y]$. We see that $I''_\beta \circ I'^{-1}_\beta = I''_\delta \circ I'^{-1}_\delta$, so the two associate the same representatives. It follows that the quotient projection Γ_{11} is given by $(\iota, \mathrm{Id}_{G_1})$ as proposed.

- We have $\Gamma_{21} : F^2E[y] \oplus L_2 \to L_2$ given by $(\iota', \mathrm{Id}_{L_2})$.
  Here the map $\iota' : F^2E[y] \hookrightarrow L_2$ is the inclusion of the third summand as written in Prop. 3.15.

  - $I'_\beta : F^2E[y]_G G_1 \xrightarrow{\sim} F^2E[y]$ given as the inverse of $\big(ffe \mapsto ffe \otimes 1_{G_1}\big)$,
  - $I''_\beta : F^2E[y]_G G_1 \xrightarrow{\iota' \otimes G_1} (L_2)_G G_1 \cong L_2$,
  - $I'_\delta : (L_2)_G FE[y] \to F^2E[y]$ given by

$$(f', f, \rho') \otimes \bar{f}\bar{e} \mapsto (\bar{f} \circ \rho) \otimes \bar{e}$$
$$= \big(\bar{f} \circ (Ef + Ef' \circ \tau + y_1\rho')\big) \otimes \bar{e}.$$

    Note that here $\bar{f}$ is interpreted as a map of complexes $\bar{f} : X_2 \to X_1$ which is composed with the map $(f', f, \rho') : R \to X_2$ to obtain a map $R \to X_1$. The composite $\bar{f} \circ \rho$ is a map $R \to X_1$, interpreted again in $F^2[y]$ according to Prop. 3.6. (So, coincidentally, the notation '$\bar{f} \circ \rho$' has two valid interpretations: one as a map $R \to X_1$, and another as a map $E^2[y] \to A[y]$ represented by an element of $F^2[y]$.)
  - $I''_\delta : (L_2)_G FE[y] \xrightarrow{L_2 \otimes \iota} (L_2)_G G_1 \cong L_2$.

  Consider the first two maps. We have that $I''_\beta \circ I'^{-1}_\beta = \iota'$ as maps $F^2E[y] \to L_2$. Consider the last two maps. One may check that $\iota' \circ I'_\delta = I''_\delta$. It follows that $\mathrm{Im}(I'_\delta - I''_\delta) \subset \mathrm{Im}(I'_\beta - I''_\beta)$, so in the quotient every element of $F^2E[y]$ is associated to a unique element of L_2, given by applying the map ι'.

- We have $\Gamma_{12} : FE^2[y] \oplus G_2 \to G_2$ given by $(\iota'', \mathrm{Id}_{G_2})$.
  Here the map $\iota'' : FE^2[y] \hookrightarrow G_2$ is the inclusion of the third summand as written in Prop. 3.15.

  - $I'_\beta : FE[y]_G G_2 \to FE^2[y]$ given by

$$\bar{f}\bar{e} \otimes (e', e, \xi') \mapsto \bar{f} \otimes (y_1y_2)^{-1}\xi(y_1\bar{e})$$
$$= \bar{f} \otimes \big(\tau(\bar{e} \otimes e) - y_2\tau(\bar{e} \otimes e') + \xi'(y_1\bar{e})\big).$$

    The map is given by considering $\bar{e}$ as a map of complexes $X_1 \to X_2$, and (e', e, ξ') as a map of complexes $X_2 \to R$, and then composing, and translating the result to bimodule form (removing y_1y_2). The final expression is computed by plugging $y_1\bar{e}$ into

$$\xi = \tau y_1(\_ \otimes e) - y_2\tau y_1(\_ \otimes e') + y_1y_2\xi'$$

from the condition of Def. 3.7, and we obtain:

$$\begin{aligned}\xi(y_1\bar{e}) &= \tau y_1(y_1\bar{e}\otimes e) - y_2\tau y_1(y_1\bar{e}\otimes e') + y_1y_2\xi' \\ &= \tau y_1y_2(\bar{e}\otimes e) - y_2\tau y_1y_2(\bar{e}\otimes e') + y_1y_2\xi' \\ &= y_1y_2\left(\tau(\bar{e}\otimes e) - y_2\tau(\bar{e}\otimes e') + \xi'\right).\end{aligned}$$

– $I''_\beta : FE[y]_G G_2 \xrightarrow{\iota\otimes G_2} (L_1)_G G_2 \cong G_2$,
– $I'_\delta : (G_1)_G FE^2[y] \xrightarrow{\sim} FE^2[y]$ given as the inverse of $\left(fee \mapsto 1_{G_1}\otimes fee\right)$,
– $I''_\delta : (G_1)_G FE^2[y] \xrightarrow{G_1\otimes\iota''} (L_1)_G G_2 \cong G_2$.

Consider the last two maps. We have that $I''_\delta \circ I'^{-1}_\delta = \iota''$ as maps $FE^2[y] \to G_2$. Now consider the first two maps. Observe that $I''_\beta = \iota'' \circ I'_\beta$. It follows that $\mathrm{Im}(I'_\beta - I''_\beta) \subset \mathrm{Im}(I'_\delta - I''_\delta)$, so every element of $FE^2[y]$ is associated in the quotient to a unique element of G_2 by applying the map ι''.

- We have $\Gamma_{22} : F^2E^2[y] \oplus (L_2)_G G_2 \to U$ given by $(\iota''', \mathrm{Id}_U)$.
  Here the map $\iota''' : F^2E^2[y] \to U$ is the inclusion of the fifth summand as written in Prop. 3.15.

– $I'_\beta : F^2E[y]_G G_2 \to F^2E^2[y]$ given by $\overline{ff}\bar{e}\otimes(e', e, \xi') \mapsto \overline{ff}\otimes(y_1y_2)^{-1}\xi(y_1\bar{e})$,
– $I''_\beta : F^2E[y]_G G_2 \xrightarrow{\iota'\otimes G_2} (L_2)_G G_2 \xrightarrow{\sim} U$ (using Lemma 3.14),
– $I'_\delta : (L_2)_G FE^2[y] \to F^2E^2[y]$ given by $(f', f, \rho')\otimes \bar{f}\overline{ee} \mapsto (\bar{f}\circ\rho)\otimes\overline{ee}$,
– $I''_\delta : (L_2)_G FE^2[y] \xrightarrow{L_2\otimes\iota''} (L_2)_G G_2 \xrightarrow{\sim} U$.

Consider the first two maps. Observe that

$$I'_\beta\left(\overline{ff}\bar{e}\otimes(e, y_1e, \xi' = 0)\right) = \overline{ff}\otimes(\bar{e}\otimes e) \in F^2E^2[y].$$

It follows that I'_β is surjective. Now we show that $\iota'''\circ I'_\beta = I''_\beta$ and that $\iota'''\circ I'_\delta = I''_\delta$ using the bimodule forms. First apply ι''' to the image under I'_β of an arbitrary simple tensor $\overline{ff}\bar{e}\otimes(e', e, \xi') \in F^2E[y]_G G_2$:

$$\iota'''\left(\overline{ff}\otimes(y_1y_2)^{-1}\xi(y_1\bar{e})\right) = \left(0,0,0,0,\Lambda^\circ = \overline{ff}\otimes(y_1y_2)^{-1}\xi(y_1\bar{e})\right),$$

then apply I''_β to the same arbitrary simple tensor, and view the result through the isomorphism $(L_2)_G G_2 \xrightarrow{\sim} U$:

$$\begin{aligned}I''_\beta\left(\overline{ff}\bar{e}\otimes(e', e, \xi')\right) &= (0,0,\overline{ff}\bar{e})\otimes_{G_1^{\mathsf{op}}}(e', e, \xi') \\ &\mapsto \left(0,0,0,0,\overline{ff}\otimes(y_1y_2)^{-1}\xi(y_1\bar{e})\right) \in U.\end{aligned}$$

So $\iota'''\circ I'_\beta = I''_\beta$. Repeat the procedure with the second pair of maps:

$$\begin{aligned}\iota'''\left((\bar{f}\circ\rho)\otimes\overline{ee}\right) &= \left(0,0,0,0,(\bar{f}\circ\rho)\otimes\overline{ee}\right), \\ I''_\delta\left((f', f, \rho')\otimes\bar{f}\overline{ee}\right) &= (f', f, \rho')\otimes_{G_1^{\mathsf{op}}}(0,0,\bar{f}\overline{ee}) \\ &\mapsto \left(0,0,0,0,(\bar{f}\circ\rho)\otimes\overline{ee}\right),\end{aligned}$$

so $\iota'''\circ I'_\delta = I''_\delta$. It follows that every element of $F^2E^2[y]$ is associated in the quotient to a unique representative in U by applying ι'''.

Remark 5.1. The map ι''' describes the inclusion of the morphisms of $\mathrm{Hom}_{K^b(B)}(R,R)$ that factor through X_1. The maps I'_β and I'_δ are in fact isomorphisms:

$$\mathrm{Hom}_{K^b(B)}(X_1,X_2)\otimes_{G_1^{\mathsf{op}}}\mathrm{Hom}_{K^b(B)}(X_2,R)\xrightarrow{\sim}\mathrm{Hom}_{K^b(B)}(X_1,R),$$
$$\mathrm{Hom}_{K^b(B)}(R,X_2)\otimes_{G_1^{\mathsf{op}}}\mathrm{Hom}_{K^b(B)}(X_2,X_1)\xrightarrow{\sim}\mathrm{Hom}_{K^b(B)}(R,X_1).$$

They are produced by reasoning as in Lemma 3.14, using that R is a finite direct sum of summands of X_2.

5.1.3. $\tilde{E}\tilde{F}$. We do not have a matrix presentation of the components of the product $\tilde{E}\tilde{F}$ from the Rickard equivalence. Instead, in this section, we proceed by studying the quotient directly, by components, determining the structure of the quotient itself, as well as the quotient projection Γ from the tensor product over Δ to the tensor product over C.

As before, in each bulleted section we propose a component of Γ. Here the arguments following a bulleted line also must justify the structure of the codomain of the Γ component written in that bulleted line. The domains are known, and in each case the annihilated submodule $\mathrm{Im}(I'_\beta - I''_\beta)+\mathrm{Im}(I'_\delta - I''_\delta)$ is defined already. Our method is to write down a map called Γ_{ij} from the appropriate domain, show that it is surjective, and show that its kernel is $\mathrm{Im}(I'_\beta - I''_\beta)+\mathrm{Im}(I'_\delta - I''_\delta)$. The codomain of Γ can be summarized in a matrix:

$$\begin{aligned}\tilde{E}\otimes_\Delta\tilde{F} &\cong \begin{pmatrix}E[y] & E^2[y]\\ G_1 & G_2\end{pmatrix}\otimes_\Delta\begin{pmatrix}F[y] & L_1\\ F^2[y] & L_2\end{pmatrix}\\ &\cong \begin{pmatrix}EF[y]\oplus E^2[y]_GF^2[y] & E[y]G_1\oplus E^2[y]_GL_2\\ G_1F[y]\oplus (G_2)_GF^2[y] & G_1G_1\oplus (G_2)_GL_2\end{pmatrix}\\ &\xrightarrow{\Gamma}\begin{pmatrix}EF[y] & E[y]G_1\\ G_1F[y] & G_1G_1\oplus EF[y]\end{pmatrix}.\end{aligned}\tag{5.4}$$

- We have $\Gamma_{11}: EF[y]\oplus E^2[y]_GF^2[y]\to EF[y]$ given by $(\mathrm{Id}_{EF[y]},\omega)$.

  Define a map $\omega: E^2[y]\otimes_{A[y]}F^2[y]\to EF[y]$ by:
  $$e_1e_2\otimes f_2f_1\mapsto e_1.f_2(y_1e_2)\otimes f_1 = e_1\otimes f_2(y_1e_2).f_1.$$

  Let $\varphi_1\in FE[y]$ be given in the second summand of (the bimodule form) $G_1^{\mathsf{op}}\cong A[y]\oplus FE[y]$. Observe that $\big(e_1\otimes\varphi_1(y_1e_2)\big)\otimes f_2f_1$ and $e_1e_2\otimes\big((f_2\circ y_1\varphi_1)\otimes f_1\big)$ are both sent by ω to $e_1.(f_2\circ y_1\varphi_1)(y_1e_2)\otimes f_1$. This means ω is middle-linear over generators in both summands of G_1^{op}, so it descends to a map, also called ω, from the tensor product $E^2[y]_GF^2[y]$ taken over G_1^{op}.

  – $I'_\beta: EE[y]_GF^2[y]\to EF[y]$ given (using bimodule forms) by
  $$\begin{aligned}e_1\otimes e_2\otimes f_2f_1 &\mapsto e_1\otimes\big(f_1\circ Ef_2\circ(e_2,y_1e_2,0)\big)\\ &= e_1\otimes\big(f_1\circ Ef_2\circ(\_\otimes y_1e_2)\big)\\ &= e_1\otimes f_1(\_.f_2(y_1e_2))\\ &= e_1\otimes f_2(y_1e_2).f_1.\end{aligned}$$

In this calculation, e_2 is interpreted as a map of complexes $X_1 \to X_2$ which induces a map $X_2 \to R$ (from $E'X_1 \to E'X_2$), precomposition with which gives the left action of $e_2 \in E[y] \subset C$ on $F^2[y] \subset \tilde{F}$. The induced map corresponds to $(e_2, y_1e_2, 0)$ in G_2 (strictly speaking, in $\bar{G}_2'$) in bimodule form. Further, $f_2f_1 \in F^2[y]$ is interpreted as a map $R \to X_1$, which is identified by $E^2[y] \xrightarrow{Ef_2} E[y] \xrightarrow{f_1} A[y]$ applied to $E^2[y]$ in the top row of R_0. The composite $X_2 \to X_1$ is identified (in the second row) by the morphism $f_1 \circ Ef_2 \circ ({}_- \otimes y_1e_2) : E[y] \to A[y]$, which is evaluated in the third and fourth lines.

– $I''_\beta : EE[y]_GF^2[y] \xrightarrow{\mathrm{Id}} E^2[y]_GF^2[y]$,
– $I'_\delta : E^2[y]_GFF[y] \to EF[y]$ given (using bimodule forms) by

$$\begin{aligned} e_1e_2 \otimes f_2 \otimes f_1 &\mapsto \big((0, f_2, 0) \circ e_1e_2\big) \otimes f_1 \\ &= y_1^{-1}(Ef_2)(y_1y_2(e_1e_2)) \otimes f_1 \\ &= e_1.f_2(y_1e_2) \otimes f_1. \end{aligned}$$

Here $f_2 : X_2 \to X_1$ induces $(0, f_2, 0) : R \to X_2$ in L_2. Further, $e_1e_2 : X_1 \to R$, and the composite map $X_1 \to X_2$ is identified by applying Ef_2 to the top row of R_0 after putting $y_1y_2e_1e_2$ in that term, and removing the final y_1 to obtain the bimodule form.

– $I''_\delta : E^2[y]_GFF[y] \xrightarrow{\mathrm{Id}} E^2[y]_GF^2[y]$.

We see that $I'_\beta = \omega$ and $I'_\delta = \omega$ after identifying $EE[y] \cong E^2[y]$ and $FF[y] \cong F^2[y]$. It follows that the kernel of Γ_{11} is the image of $I'_\beta - I''_\beta$, which is also the image of $I'_\delta - I''_\delta$, and thus $\ker(\Gamma_{11}) = \mathrm{Im}(I'_\beta - I''_\beta) + \mathrm{Im}(I'_\delta - I''_\delta)$ as desired.

Remark 5.2. The map ω corresponds on the models to the map given by composition:

$$\mathrm{Hom}_{K^b(B)}(X_2, R) \otimes_{G_1^{\mathrm{op}}} \mathrm{Hom}_{K^b(B)}(R, X_2) \to \mathrm{Hom}_{K^b(B)}(X_2, X_2).$$

- We have $\Gamma_{21} : G_1F[y] \oplus (G_2)_GF^2[y] \to G_1F[y]$ given by $(\mathrm{Id}_{G_1F[y]}, \omega')$.

Let $\omega' : (G_2)_GF^2[y] \to G_1F[y]$ be defined (using bimodule forms) by

$$\begin{aligned} (e', e, \xi') \otimes f_2f_1 &\mapsto \big((0, f_2, 0) \circ (e', e, \xi')\big) \otimes f_1 \\ &= \Big(f_2(e), y_1^{-1}Ef_2 \circ \big(y_2\tau({}_- \otimes (e - y_1e')) + y_1y_2\xi'\big)\Big) \otimes f_1 \\ &= \Big(f_2(e), Ef_2 \circ \tau({}_- \otimes (e - y_1e')) + E(f_2 \circ y_1) \circ \xi'\Big) \otimes f_1. \end{aligned}$$

Here $f_2 : X_2 \to X_1$ again induces $(0, f_2, 0) : R \to X_2$ in L_2. The composite

$$X_2 \xrightarrow{(e',e,\xi')} R \xrightarrow{(0,f_2,0)} X_2$$

is a map $X_2 \to X_2$ identified by the element of G_1 given in the next line.

– $I'_\beta : G_1E[y]_GF^2[y] \to G_1F[y]$ given (using bimodule form) by

$$\begin{aligned} ge \otimes f_2f_1 &\mapsto g \otimes \big((f_1 \circ Ef_2) \circ (e, y_1e, 0)\big) \\ &= g \otimes f_2(y_1e).f_1. \end{aligned}$$

The map $E[y]_GF^2[y] \to F[y]$ used here is the same as the one in I'_β of Γ_{11} above.

– $I''_\beta : G_1E[y]_GF^2[y] \to (G_2)_GF^2[y]$ given (using bimodule forms) by

$$\begin{aligned} (\theta, \varphi_1) \otimes e \otimes f_2f_1 &\mapsto \big((e, y_1e, 0) \circ (\theta, \varphi_1)\big) \otimes f_2f_1 \\ &= (\theta e, \theta y_1e, \varphi_1(-) \otimes e) \otimes f_2f_1. \end{aligned}$$

The map $G_1E[y] \to G_2$ used here is the same as the one in Γ_{21} for $\tilde{E}\tilde{E}$.

– $I'_\delta : (G_2)_GFF[y] \to G_1F[y]$ given by the map ω' (after identifying $FF[y]$ with $F^2[y]$),
– $I''_\delta : (G_2)_GFF[y] \xrightarrow{\text{Id}} (G_2)_GF^2[y]$.

We show that $\omega' \circ I''_\beta = I'_\beta$:

$$\begin{aligned} \omega'\big((\theta e, \theta y_1e, \varphi_1 \otimes e) \otimes f_2f_1\big) &= \big(f_2(\theta y_1e), E(f_2 \circ y_1) \circ (\varphi_1 \otimes e)\big) \otimes f_1 \\ &= \big(\theta f_2(y_1e), \varphi_1.f_2(y_1e)\big) \otimes f_1 \\ &= (\theta, \varphi_1).f_2(y_1e) \otimes f_1 \\ &= I'_\beta\big((\theta, \varphi_1) \otimes e \otimes f_2f_1\big). \end{aligned}$$

In the first line, note that $(\theta y_1e) - y_1(\theta e) = 0$ so the term '$Ef_2 \circ \tau(\_ \otimes (e - y_1e'))$' in the image under ω' disappears. Then $f_2 \circ y_1$ applied to e produces $f_2(y_1e) \in A[y]$, which acts on the right on φ_1 for the second line. For the third line, the element $f_2(y_1e) \in A[y]$ acts on G_1 on the right diagonally. It follows that $I'_\beta - I''_\beta = (\omega' - \text{Id})I''_\beta$, and therefore $\text{Im}(I'_\beta - I''_\beta) \subset \text{Im}(I'_\delta - I''_\delta)$. Thus $\ker(\Gamma_{21}) = \text{Im}(I'_\beta - I''_\beta) + \text{Im}(I'_\delta - I''_\delta)$, as desired.

- We have $\Gamma_{12} : E[y]G_1 \oplus E^2[y]_GL_2 \to E[y]G_1$ given by $(\text{Id}_{E[y]G_1}, \omega'')$.

  Let $\omega'' : E^2[y]_GL_2 \to E[y]G_1$ be defined (using bimodule forms) by

$$\begin{aligned} e_1e_2 \otimes (f', f, \rho') &\mapsto e_1 \otimes \big((f', f, \rho') \circ (e_2, y_1e_2, 0)\big) \\ &= e_1 \otimes \big(f(y_1e_2) + f'(e_2), Ef' \circ \tau(\_ \otimes e_2) + \rho'(\_ \otimes y_1e_2)\big). \end{aligned}$$

– $I'_\beta : EE[y]_GL_2 \to E[y]G_1$ given by the map ω'' (after identifying $EE[y]$ with $E^2[y]$),
– $I''_\beta : EE[y]_GL_2 \xrightarrow{\text{Id}} E^2[y]_GL_2$,
– $I'_\delta : E^2[y]_GF[y]G_1 \to E[y]G_1$ given (borrowing from I'_δ of Γ_{11}) by

$$e_1e_2 \otimes f_2 \otimes g \mapsto e_1 \otimes f_2(y_1e_2).g,$$

– $I''_\delta : E^2[y]_G F[y] G_1 \to E^2[y]_G L_2$ given (using bimodule forms) by

$$\begin{aligned} e_1e_2 \otimes f \otimes (\theta, \varphi_1) &\mapsto e_1e_2 \otimes \big((\theta,\varphi_1)\circ(0,f,0)\big) \\ &= e_1e_2 \otimes (0, f.\theta, f\otimes\varphi_1). \end{aligned}$$

Here $f : X_2 \to X_1$ induces $(0,f,0) : R \to X_2$, and the reader may check the composition with $(\theta,\varphi_1) : X_2 \to X_2$.

We show that $\omega'' \circ I''_\delta = I'_\delta$:

$$\begin{aligned} \omega''\big(e_1e_2\otimes(0,f.\theta,\varphi_1\circ Ef)\big) &= e_1\otimes\big(f(y_1e_2).\theta, (\varphi_1\circ Ef)(\_\otimes y_1e_2)\big) \\ &= e_1\otimes\big(f(y_1e_2).\theta, \varphi_1(\_.f(y_1e_2))\big) \\ &= e_1\otimes\big(f(y_1e_2).\theta, f(y_1e_2).\varphi_1\big) \\ &= e_1\otimes f(y_1e_2).(\theta,\varphi_1) \\ &= I'_\delta\big(e_1e_2\otimes f\otimes(\theta,\varphi_1)\big). \end{aligned}$$

Thus $I'_\delta - I''_\delta = (\omega'' - \mathrm{Id})I''_\delta$, and therefore $\mathrm{Im}(I'_\delta - I''_\delta) \subset \mathrm{Im}(I'_\beta - I''_\beta)$. It follows that $\ker(\Gamma_{12}) = \mathrm{Im}(I'_\beta - I''_\beta) + \mathrm{Im}(I'_\delta - I''_\delta)$, as desired.

- We have $\Gamma_{22} : G_1G_1 \oplus (G_2)_G L_2 \to G_1G_1 \oplus EF[y]$ given by $\begin{pmatrix} \mathrm{Id}_{G_1G_1} & \omega''' \\ 0 & \kappa \end{pmatrix}$.

  Below we describe the maps I'_β, I''_β, I'_δ, I''_δ, and define a map $\omega''' : (G_2)_G L_2 \to G_1G_1$, and we show that $\omega''' \circ I''_\beta = I'_\beta$ and $\omega''' \circ I''_\delta = I'_\delta$. Then we describe a decomposition of $(G_2)_G L_2$ into $(A[y], A[y])$-sub-bimodules $(G_2)_G L_2 \cong H \oplus EF[y]$ where $H = \mathrm{Im}(I''_\beta) + \mathrm{Im}(I''_\delta)$. The projection onto $EF[y]$ is called κ. (This copy of $EF[y]$ lies in the kernel of ω'''.) From all this it follows that $\ker(\Gamma_{22}) = \mathrm{Im}(I'_\beta - I''_\beta) + \mathrm{Im}(I'_\delta - I''_\delta)$ and Γ_{22} describes the projection to the quotient.

– $I'_\beta : G_1 E[y]_G L_2 \to G_1G_1$ given (borrowing from I'_β of Γ_{12}) by

$$\begin{aligned} &g\otimes e\otimes(f',f,\rho') \mapsto \\ &\quad g\otimes\big(f'(e)+f(y_1e), Ef'\circ\tau(\_\otimes e)+\rho'(\_\otimes y_1e)\big), \end{aligned}$$

– $I''_\beta : G_1 E[y]_G L_2 \to (G_2)_G L_2$ given (borrowing from I''_β of Γ_{21}) by

$$(\theta,\varphi_1)\otimes e\otimes \ell \mapsto (\theta e, \theta y_1 e, \varphi_1(-)\otimes e)\otimes \ell,$$

– $I'_\delta : (G_2)_G F[y] G_1 \to G_1G_1$ given (borrowing from I'_δ of Γ_{21}) by

$$\begin{aligned} &(e', e, \xi')\otimes f\otimes g \mapsto \\ &\quad \big(f(e), Ef\circ\tau(\_\otimes(e-y_1e')) + E(f\circ y_1)\circ\xi'\big)\otimes g, \end{aligned}$$

– $I''_\delta : (G_2)_G F[y] G_1 \to (G_2)_G L_2$ given (borrowing from I''_δ of Γ_{12}) by

$$g\otimes f\otimes(\theta,\varphi_1) \mapsto g\otimes(0, f.\theta, f\otimes\varphi_1).$$

Now we define a morphism of $(A[y], A[y])$-bimodules $\omega''' : G_2 \otimes_{A[y]} L_2 \to G_1G_1$, and in a subsequent lemma we show that ω''' descends to a morphism $\omega''' : G_2 \otimes_{G_1^{\mathsf{op}}} L_2 \to G_1G_1$ by showing that it is also middle-linear over generators of G_1^{op} in $FE[y]$. (Since $G_1 \cong A[y] \oplus FE[y]$ as a bimodule, this ensures linearity over all of G_1^{op}.) Let $(e', e, \xi') \otimes (f', f, \rho') \in G_2 \otimes_{A[y]} L_2$ be an arbitrary simple tensor. We define:

$$\begin{aligned}\omega''' : (e', e, \xi') \otimes (f', f, \rho') \mapsto \Big(&\varepsilon(e' \otimes f') + \varepsilon(e \otimes f), FE(\varepsilon \circ y_1 F)(\xi' \otimes f) \\ &+ FE\varepsilon(\xi' \otimes f') + \sigma(e \otimes f) - \sigma(y_1 e' \otimes f)\Big) \otimes (1, 0) \\ &+ (1, 0) \otimes \Big(0, \varepsilon FE(e \otimes \rho') + \sigma(e' \otimes f')\Big) + \sigma FE(e \otimes \rho') \\ &- \sigma FE(y_1 e' \otimes \rho') + FE\sigma(\xi' \otimes f') + FE(\varepsilon \circ y_1 F)FE(\xi' \otimes \rho').\end{aligned}$$

The last four terms, beginning with $\sigma FE(e \otimes \rho')$, are elements of $FEFE[y]$. They should be interpreted in the last summand of G_1G_1 that appears in the following decomposition of bimodules:

$$\begin{aligned} G_1 \otimes_{A[y]} G_1 &\xrightarrow{\sim} A[y] \oplus FE[y] \oplus FE[y] \oplus FEFE[y], \\ (\theta, \varphi_1) \otimes (\theta', \varphi_1') &\mapsto \big(\theta\theta', \theta.\varphi_1', \varphi_1.\theta', \varphi_1 \otimes \varphi_1'\big). \end{aligned} \tag{5.5}$$

At this point ω''' has been defined as a map $G_2 \otimes_{A[y]} L_2 \to G_1G_1$. It is useful to go further and record the data of ω''' as a matrix. We can give a decomposition of $G_2 \otimes_{A[y]} L_2$ into a direct sum of $(A[y], A[y])$-bimodules:

$$\begin{aligned} &G_2 \otimes_{A[y]} L_2 \xrightarrow{\sim} EF[y]^{\oplus 4} \oplus FE^2F[y]^{\oplus 2} \oplus EF^2E[y]^{\oplus 2} \oplus FE^2F^2E[y], \\ &(e', e, \xi') \otimes (f', f, \rho') \mapsto (e' \otimes f', e' \otimes f, e \otimes f', e \otimes f) \\ &\qquad\qquad \oplus (e' \otimes \rho', e \otimes \rho') \oplus (\xi' \otimes f', \xi' \otimes f) \oplus (\xi' \otimes \rho'). \end{aligned}$$

Each of the terms in the formula for ω''' is a morphism of $(A[y], A[y])$-bimodules.

Definition 5.3. Using the ordered decompositions of $G_2 \otimes_{A[y]} L_2$ and of G_1G_1 above, the map $\omega''' : G_2 \otimes_{A[y]} L_2 \to G_1G_1$ is given by the following matrix:

$$\begin{pmatrix} \varepsilon & 0 & 0 & \varepsilon & 0 & 0 & 0 & 0 & 0 \\ \sigma & 0 & 0 & 0 & 0 & \varepsilon FE & 0 & 0 & 0 \\ 0 & -\sigma \circ y_1 F & 0 & \sigma & 0 & 0 & FE\varepsilon & FE(\varepsilon \circ y_1 F) & 0 \\ 0 & 0 & 0 & 0 & -(\sigma \circ y_1 F)FE & \sigma FE & FE\sigma & 0 & FE(\varepsilon \circ y_1 F)FE \end{pmatrix}.$$

Lemma 5.4. *The map ω''' is middle-linear over the action of generators of the summand $FE[y] \subset G_1^{\mathsf{op}}$.*

Proof. We first compute the middle actions $(e', e, \xi').\varphi_1$ and $\varphi_1.(f', f, \rho')$ for $\varphi_1 \in FE[y] \subset G_1^{\mathsf{op}}$, $(e', e, \xi') \in G_2$, and $(f', f, \rho') \in L_2$, both in bimodule form. These are:

$$\begin{aligned} (e', e, \xi').\varphi_1 &= \big(\varphi_1(e), y_1\varphi_1(e), E\varphi_1 \circ \tau(\_ \otimes (e - y_1 e')) + E(\varphi_1 y_1) \circ \xi'\big) \\ \varphi_1.(f', f, \rho') &= \big(0, f \circ y_1\varphi_1 + f' \circ \varphi_1, Ef' \circ \tau \circ E\varphi_1 + \rho' \circ E(y_1\varphi_1)\big). \end{aligned}$$

Using the formulas above, one easily computes the images under ω''' of $(e', e, \xi').\varphi_1 \otimes (f', f, \rho')$ and $(e', e, \xi') \otimes \varphi_1.(f', f, \rho')$ and checks that they agree. □

Corollary 5.5. *It follows from Lemma 5.4 that ω''' determines a morphism of $(A[y], A[y])$-bimodules $\omega''' : (G_2)_G L_2 \to G_1 G_1$.*

We show next that $\omega''' \circ I''_\beta = I'_\beta$ and $\omega''' \circ I''_\delta = I'_\delta$. The formula for ω''' is determined by these conditions and may be derived from them. Evaluating the left side of the first equation:

$$\begin{aligned}
&\omega''' \circ I''_\beta\big((\theta, \varphi_1) \otimes e \otimes (f', f, \rho')\big)\\
&= \omega'''\big((\theta e, \theta y_1 e, \varphi_1 \otimes e) \otimes (f', f, \rho')\big)\\
&= \big(f'(\theta e) + f(\theta y_1 e), \varphi_1.f(y_1 e) + \varphi_1.f'(e)\big) \otimes (1, 0)\\
&\quad + (1, 0) \otimes \big(0, \rho'(_ \otimes \theta y_1 e) + Ef' \circ \tau(_ \otimes \theta e)\big)\\
&\quad + (0, \varphi_1) \otimes \big(0, Ef' \circ \tau(_ \otimes e) + \rho'(_ \otimes y_1 e)\big)\\
&= \big(\theta.(f'(e) + f(y_1 e)), \varphi_1.(f'(e) + f(y_1 e))\big) \otimes (1, 0)\\
&\quad + (\theta, \varphi_1) \otimes \big(0, Ef' \circ \tau(_ \otimes e) + \rho'(_ \otimes y_1 e)\big)\\
&= (\theta, \varphi_1) \otimes \big(f'(e) + f(y_1 e), Ef' \circ \tau(_ \otimes e) + \rho'(_ \otimes y_1 e)\big)\\
&= I'_\beta\big((\theta, \varphi_1) \otimes e \otimes (f', f, \rho')\big).
\end{aligned}$$

Now evaluating the left side of the second equation:

$$\begin{aligned}
&\omega''' \circ I''_\delta\big((e', e, \xi') \otimes f \otimes (\theta, \varphi_1)\big)\\
&= \omega'''\big((e', e, \xi') \otimes (0, f.\theta, f \otimes \varphi_1)\big)\\
&= \big(f(e).\theta, E(f.\theta \circ y_1) \circ \xi' + E(f.\theta) \circ \tau(_ \otimes (e - y_1 e'))\big) \otimes (1, 0)\\
&\quad + (1, 0) \otimes \big(0, f(e).\varphi_1\big) + \big(0, Ef \circ \tau(_ \otimes (e - y_1 e')\big) \otimes (0, \varphi_1)\\
&\quad + \big(0, E(f \circ y_1) \circ \xi'\big) \otimes (0, \varphi_1)\\
&= \big(f(e), E(f \circ y_1) \circ \xi' + Ef \circ \tau(_ \otimes (e - y_1 e'))\big) \otimes (\theta, 0)\\
&\quad + \big(f(e), Ef \circ \tau(_ \otimes (e - y_1 e')) + E(f \circ y_1) \circ \xi'\big) \otimes (0, \varphi_1)\\
&= \big(f(e), Ef \circ \tau(_ \otimes (e - y_1 e')) + E(f \circ y_1) \circ \xi'\big) \otimes (\theta, \varphi_1)\\
&= I'_\delta\big((e', e, \xi') \otimes f \otimes (\theta, \varphi_1)\big).
\end{aligned}$$

Now the product $(G_2)_G L_2$ is the quotient of the product $(G_2)_{A[y]} L_2$ by the image of $\gamma' - \gamma''$, where:

- $\gamma' : \big(G_2 \otimes_{A[y]} FE[y]\big) \otimes_{A[y]} L_2 \to (G_2)_{A[y]} L_2$ given by
  $(e', e, \xi') \otimes \varphi_1 \otimes \ell \mapsto$
  $\big(\varphi_1(e), y_1\varphi_1(e), E\varphi_1 \circ \tau(\_ \otimes (e - y_1 e')) + E(\varphi_1 \circ y_1) \circ \xi'\big) \otimes \ell$,
- $\gamma'' : G_2 \otimes_{A[y]} \big(FE[y] \otimes_{A[y]} L_2\big) \to (G_2)_{A[y]} L_2$ given by
  $g \otimes \varphi_1 \otimes (f', f, \rho') \mapsto$
  $g \otimes \big(0, f' \circ \varphi_1 + f \circ y_1\varphi_1, Ef' \circ \tau \circ E\varphi_1 + \rho' \circ E(y_1\varphi_1)\big)$.

There is a copy of $EF[y]$ in $(G_2)_{A[y]}L_2$ generated by terms of the form $(0,e,0)\otimes(f',0,0)$. Let $\bar{H}$ be its direct complement. The images of γ' and γ'' lie in $\bar{H}$, so $(G_2)_G L_2 \cong H \oplus EF[y]$, where H is the quotient of $\bar{H}$ by the image of $\gamma' - \gamma''$.

The image of I''_β includes every term of the form $(e, y_1 e, \varphi_1 \otimes e)\otimes \ell$, and the image of I''_δ includes every term of the form $g\otimes(0,f,f\otimes\varphi_1)$. By adding appropriate linear combinations of terms of the first form, one obtains any element $(e, y_1e, \xi')\otimes\ell$, and similarly from terms of the second form one obtains any $g\otimes(0,f,\rho')$. It follows that $\mathrm{Im}(I''_\beta + I''_\delta) = H$.

5.2. **Maps $\tilde{\rho}_\lambda$: formulas.** In this section we derive formulas by matrix components for the maps $\tilde{\sigma} = \tilde{F}\tilde{E}\tilde{\varepsilon} \circ \tilde{F}\tilde{\tau}\tilde{F} \circ \tilde{\eta}\tilde{E}\tilde{F}$, $\tilde{\varepsilon}\circ\tilde{x}^i\tilde{F}$, and $\tilde{F}\tilde{x}^i \circ \tilde{\eta}$ that are used to define the maps $\tilde{\rho}_\lambda$. We will be using the matrix components for $\tilde{E}$, $\tilde{F}$, $\tilde{E}\tilde{E}$, $\tilde{F}\tilde{E}$, and $\tilde{E}\tilde{F}$ that were found and studied in previous sections. (See Eqs. (2.4), (4.1), (2.5), (5.2), and (5.4), respectively.) The unit and counit $\tilde{\eta}$ and $\tilde{\varepsilon}$ are given by the duality pairing and thus are easily interpreted in terms of maps between complexes where that is convenient. The morphisms $\tilde{x}$ and $\tilde{\tau}$ were given on components in Eqs. (2.6) and (2.7).

5.2.1. *Map $\tilde{\sigma} : \tilde{E}\tilde{F} \to \tilde{F}\tilde{E}$.* We begin by computing the map $\tilde{\sigma} : \tilde{E}\tilde{F} \to \tilde{F}\tilde{E}$. Recall that $\tilde{\sigma}$ is defined by $\tilde{\sigma} = \tilde{F}\tilde{E}\tilde{\varepsilon} \circ \tilde{F}\tilde{\tau}\tilde{F} \circ \tilde{\eta}\tilde{E}\tilde{F}$, and $\tilde{\eta}$, $\tilde{\varepsilon}$, and $\tilde{\tau}$ are determined already. We will need formulas for each component of $\tilde{\sigma}$ in its matrix presentation.

We use the following technique to derive the formulas. We start with an appropriate matrix coefficient of the element $[\tilde{\eta}(1)] \in [\tilde{F}\tilde{E}]$, together with an arbitrary generator of a component of the matrix $[\tilde{E}\tilde{F}]$. Then we write the latter as a sum of simple tensor products of elements of $[\tilde{E}]$ with elements of $[\tilde{F}]$. As a point of notation, this will be said to lie in $[\tilde{E}]\cdot[\tilde{F}]$ (and similarly for other matrix products). Then we write $[\tilde{\eta}(1)]$ in $[\tilde{F}]\cdot[\tilde{E}]$, and taking another tensor product we have an element we can write in $[\tilde{F}]\cdot[\tilde{E}]\cdot[\tilde{E}]\cdot[\tilde{F}]$. Upon this we apply $[\tilde{F}]\cdot[\tilde{\tau}]\cdot[\tilde{F}]$ using (2.7). We view the result in $[\tilde{F}]\cdot[\tilde{E}]\cdot[\tilde{E}\tilde{F}]$, apply $[\tilde{F}]\cdot[\tilde{E}]\cdot[\tilde{\varepsilon}]$ to obtain an element of $[\tilde{F}]\cdot[\tilde{E}]\cdot[C]$, view this in $[\tilde{F}\tilde{E}]\cdot[C]$, and allow the coefficient in $[C]$ to act on the right on the coefficient in $[\tilde{F}\tilde{E}]$. The result is the image under $[\tilde{\sigma}]$ of the arbitrary generator in $[\tilde{E}\tilde{F}]$ with which we began.

The following bulleted lines state the results of this procedure, and the procedure itself is carried out in detail in the paragraphs below those lines.

- We have $[\tilde{\sigma}]_{11} : [\tilde{E}\tilde{F}]_{11} \to [\tilde{F}\tilde{E}]_{11}$ given by $\left(\begin{smallmatrix}\varepsilon\\ \sigma\end{smallmatrix}\right)$ using the decompositions:
  - $[\tilde{E}\tilde{F}]_{11} \cong EF[y]$,
  - $[\tilde{F}\tilde{E}]_{11} \cong (G_1)_G G_1 \cong G_1 \cong A[y] \oplus FE[y]$.

  We take $[\tilde{\eta}(1)]_{11} = (1,0)\otimes(1,0) \in (G_1)_G G_1 \cong [\tilde{F}\tilde{E}]_{11}$ (using bimodule form), and an arbitrary generator $e\otimes f \in EF[y] \cong [\tilde{E}\tilde{F}]_{11}$. The product of these in $[\tilde{F}\tilde{E}]\cdot[\tilde{E}\tilde{F}]$ can be represented in $[\tilde{F}]\cdot[\tilde{E}]\cdot[\tilde{E}]\cdot[\tilde{F}]$ by:

$$\left(\begin{smallmatrix}0 & (1,0)\\ 0 & 0\end{smallmatrix}\right)\cdot\left(\begin{smallmatrix}0 & 0\\ (1,0) & 0\end{smallmatrix}\right)\cdot\left(\begin{smallmatrix}e & 0\\ 0 & 0\end{smallmatrix}\right)\cdot\left(\begin{smallmatrix}f & 0\\ 0 & 0\end{smallmatrix}\right) \in \begin{pmatrix}F[y] & L_1\\ F^2[y] & L_2\end{pmatrix}\cdot\begin{pmatrix}E[y] & E^2[y]\\ G_1 & G_2\end{pmatrix}\cdot\begin{pmatrix}E[y] & E^2[y]\\ G_1 & G_2\end{pmatrix}\cdot\begin{pmatrix}F[y] & L_1\\ F^2[y] & L_2\end{pmatrix}.$$

The middle factors give $(1,0) \otimes e \in G_1 \otimes_{A[y]} E[y]$. Passing through Γ_{21} of Eq. (5.1), this represents $(e, y_1 e, 0) \in G_2 \cong [\tilde{E}^2]_{21}$. To apply $[\tilde{\tau}]_{21}$ from Eq. (2.7) we translate that formula from the terms of $\bar{G}_2$ to those of $\bar{G}_2'$ in bimodule form. Using $e_1 - e_2 = y_1 e'$, we have:

$$\begin{aligned} \xi &= \_ \otimes e_1 + y_2\tau(\_ \otimes e_2) + y_1 y_2 \xi' \\ &= \_ \otimes (e_1 - e_2) + \tau y_1(\_ \otimes e_2) + y_1 y_2 \xi'; \\ \tau \circ \xi &= \tau(\_ \otimes (e_1 - e_2)) + \tau^2 y_1(\_ \otimes e_2) + \tau y_1 y_2 \xi' \\ &= \tau y_1(\_ \otimes e') + y_1 y_2 \tau \circ \xi' \\ &= \_ \otimes e' + y_2 \tau(\_ \otimes e') + y_1 y_2(\tau \circ \xi'). \end{aligned}$$

So instead of $(e_1, e_2, \xi) \overset{\tilde{\tau}}{\mapsto} (e', e', \tau \circ \xi)$ in the terms of $\bar{G}_2$, the formula is $(e', e, \xi') \overset{\tilde{\tau}}{\mapsto} (0, e', \tau \circ \xi')$ in the terms of $\bar{G}_2'$. Application of $[\tilde{\tau}]_{21}$ to $(e, y_1 e, 0)$ therefore yields $(0, e, 0)$, which may be represented in $[\tilde{E}] \cdot [\tilde{E}]$ by:

$$\begin{pmatrix} 0 & 0 \\ 0 & (0,e,0) \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ (1,0) & 0 \end{pmatrix} \in [\tilde{E}] \cdot [\tilde{E}].$$

Then:

$$\begin{pmatrix} 0 & 0 \\ (1,0) & 0 \end{pmatrix} \cdot \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} \overset{\tilde{\varepsilon}}{\longmapsto} \begin{pmatrix} 0 & 0 \\ f & 0 \end{pmatrix} \in [C]$$

and

$$\begin{pmatrix} 0 & (1,0) \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & (0,e,0) \end{pmatrix} \overset{\Gamma_{12}}{\longmapsto} \begin{pmatrix} 0 & (0,e,0) \\ 0 & 0 \end{pmatrix} \in \begin{pmatrix} G_1 & G_2 \\ L_2 & U \end{pmatrix} = [\tilde{F}\tilde{E}]$$

using Γ_{12} in Eq. (5.3). Finally letting $f \in C$ act on the right, we have:

$$\begin{pmatrix} 0 & (0,e,0) \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ f & 0 \end{pmatrix} = \begin{pmatrix} \big(f(e), Ef \circ \tau(\_ \otimes e)\big) & 0 \\ 0 & 0 \end{pmatrix} \in [\tilde{F}\tilde{E}].$$

The nonzero coefficient may be interpreted as $\big(\varepsilon(e \otimes f), \sigma(e \otimes f)\big)$.

- We have $[\tilde{\sigma}]_{21} : [\tilde{E}\tilde{F}]_{21} \to [\tilde{F}\tilde{E}]_{21}$ given by $\begin{pmatrix} 1 & 0 \\ 0 & F\varepsilon \\ 0 & F\sigma \end{pmatrix}$ using the decompositions:
  - $[\tilde{E}\tilde{F}]_{21} \cong G_1 F[y] \cong F[y] \oplus FEF[y]$,
  - $[\tilde{F}\tilde{E}]_{21} \cong L_2 \cong F[y] \oplus F[y] \oplus F^2E[y]$.

  Considering the isomorphism $FE \xrightarrow{\sim} \mathrm{Hom}_A({}_AE, E)$, we can choose an expression for $\eta(1) \in FE \subset FE[y]$ corresponding to $\mathrm{Id}_E \in \mathrm{Hom}_A({}_AE, E)$ as a sum of simple tensors:

$$\eta(1) = \sum_{a \in Q} f_a \otimes e_a \in FE \subset FE[y],$$

where Q is some finite index set. Using f_a, e_a for $a \in Q$, we find an expression for $[\tilde{\eta}(1)]_{22}$ in $(L_2)_G G_2$:

Lemma 5.6. *The element*

$$\sum_{a \in Q} (f_a, 0, 0) \otimes (e_a, 0, 0) + \sum_{b \in Q} (0, f_b, 0) \otimes (0, e_b, 0) \in (L_2)_G G_2$$

(written using bimodule forms) is sent to $\mathrm{Id}_R \in U$ *under the composition morphism* $(L_2)_G G_2 \xrightarrow{\sim} U$ *of Lemma 3.14. We write* $[\tilde{\eta}(1)]_{22}$ *for this element.*

Proof. We first take composition of the first sum, and then of the second.

Claim 5.7. Under the map $(L_2)_G G_2 \xrightarrow{\sim} U$, we have:

$$\sum_{a\in Q}(f_a,0,0)\otimes(e_a,0,0)\mapsto(0,0,0,\mathrm{Id}_{E[y]},0).$$

Proof of Claim. The matrix $[\Phi]$ giving the degree 1 bottom row part of the image, which is a morphism in $\mathrm{Hom}_{K^b(B)}(R,R)$ written in U, is $\sum_{a\in Q}\left(\begin{smallmatrix}0 & 0\\ 0 & f_a(\text{-})\otimes e_a\end{smallmatrix}\right) = \left(\begin{smallmatrix}0 & 0\\ 0 & 1\end{smallmatrix}\right)$. To compute the fifth coefficient Λ° of the image, we find the degree 0 part Λ of the map on the top row, given by taking the composition $E^2[y]\to A[y]\to E^2[y]$:

$$\begin{aligned}&\sum_{a\in Q} -y_2\tau(\_\otimes y_1e_a)\circ(Ef_a\circ\tau)\\ &=\sum_{a\in Q,d\in P} -y_2\tau y_1\big(\tau(\_\,\_)_{(1d)}\otimes f_a(\tau(\_\,\_)_{(2d)}).e_a\big)\\ &=-y_2\tau y_1\tau=-y_2\tau\end{aligned}$$

(in the second line we introduce notation for a decomposition $\tau(ee)=\sum_{d\in P}\tau(ee)_{(1d)}\otimes\tau(ee)_{(2d)}$ for some choices of $\tau(ee)_{(id)}$, $i=1,2$ and finite index set P, and in the third line we use that $\sum_{a\in Q} f_a(e^*).e_a=e^*$ for any $e^*\in E[y]$). Then $\Lambda^\circ=0$ is determined by Eq. (3.3) with this Λ and Φ. □

Claim 5.8. Under the map $(L_2)_G G_2 \xrightarrow{\sim} U$, we have:

$$\sum_{b\in Q}(0,f_b,0)\otimes(0,e_b,0)\mapsto(\mathrm{Id}_{E[y]},0,0,0,0).$$

Proof of Claim. Computing as above, the matrix $[\Phi]$ is given by $\left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)$, and we have:

$$\begin{aligned}\sum_{b\in Q}\big(\_\otimes e_b+y_2\tau(\_\otimes e_b)\big)\circ Ef_b&=\sum_{b\in Q}\tau y_1(\_\otimes e_b)\circ Ef_b\\ &=\tau y_1\Big(\sum_{b\in Q} f_b(\_).e_b\Big)=\tau y_1.\end{aligned}$$

Again, $\Lambda^\circ=0$ is determined by Eq. (3.3) with this Λ and Φ. □

So $[\tilde\eta(1)]_{22}$ is sent to $(1,0,0,1,0)\in U$, which indeed corresponds to Id_R. □

Then we take an arbitrary generator $(\theta,\varphi_1)\otimes f\in G_1F[y]\cong[\tilde E\tilde F]_{21}$. Expressing the product $\tilde\eta(1)\otimes(\theta,\varphi_1)\otimes f$ in $[\tilde F]\cdot[\tilde E]\cdot[\tilde E]\cdot[\tilde F]$, we have:

$$\begin{aligned}&\sum_{a\in Q}\left(\begin{smallmatrix}0 & 0\\ 0 & (f_a,0,0)\end{smallmatrix}\right)\cdot\left(\begin{smallmatrix}0 & 0\\ 0 & (e_a,0,0)\end{smallmatrix}\right)\cdot\left(\begin{smallmatrix}0 & 0\\ (\theta,\varphi_1) & 0\end{smallmatrix}\right)\cdot\left(\begin{smallmatrix}f & 0\\ 0 & 0\end{smallmatrix}\right)\\ &+\sum_{b\in Q}\left(\begin{smallmatrix}0 & 0\\ 0 & (0,f_b,0)\end{smallmatrix}\right)\cdot\left(\begin{smallmatrix}0 & 0\\ 0 & (0,e_b,0)\end{smallmatrix}\right)\cdot\left(\begin{smallmatrix}0 & 0\\ (\theta,\varphi_1) & 0\end{smallmatrix}\right)\cdot\left(\begin{smallmatrix}f & 0\\ 0 & 0\end{smallmatrix}\right)\\ &\in\begin{pmatrix}F[y] & L_1\\ F^2[y] & L_2\end{pmatrix}\cdot\begin{pmatrix}E[y] & E^2[y]\\ G_1 & G_2\end{pmatrix}\cdot\begin{pmatrix}E[y] & E^2[y]\\ G_1 & G_2\end{pmatrix}\cdot\begin{pmatrix}F[y] & L_1\\ F^2[y] & L_2\end{pmatrix}.\end{aligned}$$

Now we interpret $\left(\begin{smallmatrix}0 & 0\\ 0 & (e_a,0,0)\end{smallmatrix}\right)\cdot\left(\begin{smallmatrix}0 & 0\\ (\theta,\varphi_1) & 0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0 & 0\\ 0 & (0,e_b,0)\end{smallmatrix}\right)\cdot\left(\begin{smallmatrix}0 & 0\\ (\theta,\varphi_1) & 0\end{smallmatrix}\right)$ in $[\tilde E^2]$ using Γ_{21} from Eq. (5.1); this requires the right action of G_1^{op} on R from

Lemma 3.2. Then we apply $[\tilde{\tau}]$.

$$\begin{aligned}\Gamma_{21} : (e_a, 0, 0) \otimes (\theta, \varphi_1) &\mapsto (e_a, 0, 0).(\theta, \varphi_1)\\ &= \big(e_a.\theta, 0, -E\varphi_1 \circ \tau(\_ \otimes y_1 e_a)\big) \in G_2 = [\tilde{E}^2]_{21}\\ &\overset{\tilde{\tau}}{\mapsto} \big(0, e_a.\theta, -\tau \circ E\varphi_1 \circ \tau(\_ \otimes y_1 e_a)\big) \in [\tilde{E}^2]_{21},\\ \Gamma_{21} : (0, e_b, 0) \otimes (\theta, \varphi_1) &\mapsto (0, e_b, 0).(\theta, \varphi_1)\\ &= \big(\varphi_1(e_b), \varphi(e_b), E\varphi_1 \circ \tau(\_ \otimes e_b)\big) \in [\tilde{E}^2]_{21}\\ &\overset{\tilde{\tau}}{\mapsto} \big(0, \varphi_1(e_b), \tau \circ E\varphi_1 \circ \tau(\_ \otimes e_b)\big) \in [\tilde{E}^2]_{21}.\end{aligned}$$

We can represent these in $[\tilde{E}] \cdot [\tilde{E}]$ using the isomorphism $G_2 \xrightarrow{\sim} (G_2)_G G_1$, $g \mapsto g \otimes (1, 0)$. So, after applying $[\tilde{F}] \cdot [\tilde{\tau}] \cdot [\tilde{F}]$ to the middle terms, we have:

$$\begin{aligned}&\sum_{a \in Q} \begin{pmatrix} 0 & 0 \\ 0 & (f_a,0,0) \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & \big(0, e_a.\theta, -\tau\circ E\varphi_1 \circ \tau(\_\otimes y_1 e_a)\big) \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ (1,0) & 0 \end{pmatrix} \cdot \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix}\\ &+ \sum_{b \in Q} \begin{pmatrix} 0 & 0 \\ 0 & (0,f_b,0) \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & \big(0, \varphi_1(e_b), \tau\circ E\varphi_1 \circ \tau(\_\otimes e_b)\big) \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ (1,0) & 0 \end{pmatrix} \cdot \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix}.\end{aligned}$$

Then $\tilde{\varepsilon} : \begin{pmatrix} 0 & 0 \\ (1,0) & 0 \end{pmatrix} \cdot \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} \mapsto f \in F[y] \cong [C]_{21}$, so by applying $[\tilde{F}] \cdot [\tilde{E}] \cdot [\tilde{\varepsilon}]$ and viewing the first two factors in $(L_2)_G G_2 \subset [\tilde{F}\tilde{E}]_{22}$ we obtain:

$$\begin{pmatrix} 0 & 0 \\ 0 & \begin{matrix}\sum_{a\in Q}(f_a,0,0)\otimes\big(0, e_a.\theta, -\tau\circ E\varphi_1\circ\tau(\_\otimes y_1 e_a)\big)\\ +\sum_{b\in Q}(0,f_b,0)\otimes\big(0,\varphi_1(e_b),\tau\circ E\varphi_1\circ\tau(\_\otimes e_b)\big)\end{matrix} \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ f & 0 \end{pmatrix} \in [\tilde{F}\tilde{E}] \cdot [C].$$

Now we express this element in $L_2 = [\tilde{F}\tilde{E}]_{21}$ by applying the composition map $(L_2)_G G_2 \xrightarrow{\sim} U$ and then evaluating the action of $f \in [C]_{21}$ on the right. The latter may be computed by embedding f in L_2 as $(0, f, 0)$ and post-composing with this element.

Passing first through the composition map $(L_2)_G G_2 \xrightarrow{\sim} U$, we have:

$$\sum_{a \in Q} \big(0, e_a.\theta, -\tau \circ E\varphi_1 \circ \tau(\_ \otimes y_1 e_a)\big) \circ (f_a, 0, 0) \mapsto (0, 0, \theta, 0, -\tau \circ E\varphi_1 \circ \tau).$$

In the first components of this calculation, we have used:

$$\begin{aligned}\sum_{a \in Q} f_a(-).(e_a.\theta) &: E[y] \to E[y]\\ &e \mapsto \sum_{a \in Q} f_a(e).(e_a.\theta) = e.\theta,\end{aligned}$$

and for the last component we have used:

$$\begin{aligned}&\sum_{a \in Q} \big(\_ \otimes e_a.\theta + y_2 \tau(\_ \otimes e_a.\theta) - y_1 y_2 \tau \circ E\varphi_1 \circ \tau(\_ \otimes y_1 e_a)\big) \circ (Ef_a \circ \tau)\\ &= \tau.\theta - y_1 y_2 \tau \circ E\varphi_1 \circ \tau\\ &= \tau y_1 (E\theta \circ \tau) + y_1 y_2 (-\tau \circ E\varphi_1 \circ \tau).\end{aligned}$$

The fact that $\Lambda^\circ = -\tau \circ E\varphi_1 \circ \tau$ can be deduced by comparing with Eq. (3.3) where $[\Phi] = \left(\begin{smallmatrix} 0 & E\theta \\ 0 & 0 \end{smallmatrix}\right)$. Similarly, we have:

$$\sum_{b\in Q} \big(0, \varphi_1(e_b), \tau \circ E\varphi_1 \circ \tau(\_ \otimes e_b)\big) \circ (0, f_b, 0) \mapsto (\varphi_1, 0, 0, 0, \tau \circ E\varphi_1 \circ \tau),$$

where again we have used:

$$\begin{aligned} &\sum_{b\in Q} \big(\_ \otimes \varphi_1(e_b) + y_2\tau(\_ \otimes \varphi_1(e_b)) + y_1y_2\tau \circ E\varphi_1 \circ \tau(\_ \otimes e_b)\big) \circ Ef_b \\ &= \tau y_1 \circ E\varphi_1 + y_1y_2\tau \circ E\varphi_1 \circ \tau \\ &= \tau y_1(E\varphi_1) + y_1y_2(\tau \circ E\varphi_1 \circ \tau), \end{aligned}$$

so $\Lambda^\circ = \tau \circ E\varphi_1 \circ \tau$. For the sum of the images, we have $(\varphi_1, 0, \theta, 0, 0) \in U$. Next we compute the right action of $f \in [C]_{21}$ on this element:

$$(0, f, 0) \circ (\varphi_1, 0, \theta, 0, 0) = (\theta.f, f \circ \varphi_1, Ef \circ \tau \circ E\varphi_1),$$

where we have used:

$$\begin{aligned} &Ef \circ \big(\tau y_1(E\varphi_1 + E\theta \circ \tau)\big) \\ &= Ef \circ \big(\tau y_1 \circ E\varphi_1 + \tau y_1 \tau \circ E\theta\big) \\ &= Ef \circ \big(\tau y_1 \circ E\varphi_1 + \tau \circ E\theta\big) \\ &= Ef \circ \big(y_2 \tau \circ E\varphi_1 + E\varphi_1 + E\theta \circ \tau\big) \\ &= E(\theta.f) \circ \tau + E(f \circ \varphi_1) + y_1 \circ (Ef \circ \tau \circ E\varphi_1). \end{aligned}$$

Our final expression for the image of $\left(\begin{smallmatrix} 0 & 0 \\ 0 & (\theta,\varphi_1)\otimes f \end{smallmatrix}\right)$ under $[\tilde{\sigma}]_{21}$ is therefore:

$$\begin{pmatrix} 0 & 0 \\ (\theta.f, f \circ \varphi_1, Ef \circ \tau \circ E\varphi_1) & 0 \end{pmatrix} \in \begin{pmatrix} G_1 & G_2 \\ L_2 & U \end{pmatrix} = [\tilde{F}\tilde{E}].$$

The bulleted statement follows from the fact that $f \circ \varphi_1 = F\varepsilon(\varphi_1 \otimes f)$ and $Ef \circ \tau \circ E\varphi_1 = F\sigma(\varphi_1 \otimes f)$.

- We have $[\tilde{\sigma}]_{12} : [\tilde{E}\tilde{F}]_{12} \to [\tilde{F}\tilde{E}]_{12}$ given by $\left(\begin{smallmatrix} 0 & \varepsilon E \\ 1 & y_1 \circ \varepsilon E \\ 0 & \sigma E \end{smallmatrix}\right)$ using the decompositions:
  - $[\tilde{E}\tilde{F}]_{12} \cong E[y]G_1 \cong E[y] \oplus EFE[y]$,
  - $[\tilde{F}\tilde{E}]_{12} \cong G_2 \cong E[y] \oplus E[y] \oplus FE^2[y]$.

  We take $[\tilde{\eta}(1)]_{11} = (1,0) \otimes (1,0) \in G_1G_1 \cong [\tilde{F}\tilde{E}]_{11}$, and an arbitrary generator $e \otimes (\theta, \varphi_1) \in E[y]G_1 \cong [\tilde{E}\tilde{F}]_{12}$. The product of these in $[\tilde{F}\tilde{E}] \cdot [\tilde{E}\tilde{F}]$ can be expressed in $[\tilde{F}] \cdot [\tilde{E}] \cdot [\tilde{E}] \cdot [\tilde{F}]$ by:

$$\left(\begin{smallmatrix} 0 & (1,0) \\ 0 & 0 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 0 & 0 \\ (1,0) & 0 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} e & 0 \\ 0 & 0 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 0 & (\theta,\varphi_1) \\ 0 & 0 \end{smallmatrix}\right),$$

  and application of $[\tilde{F}] \cdot [\tilde{\tau}] \cdot [\tilde{F}]$ gives:

$$\left(\begin{smallmatrix} 0 & (1,0) \\ 0 & 0 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 0 & 0 \\ 0 & (0,e,0) \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 0 & 0 \\ (1,0) & 0 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 0 & (\theta,\varphi_1) \\ 0 & 0 \end{smallmatrix}\right).$$

  This is sent by $[\tilde{F}] \cdot [\tilde{E}] \cdot [\tilde{\varepsilon}]$ to

$$\left(\begin{smallmatrix} 0 & (0,e,0) \\ 0 & 0 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 0 & 0 \\ 0 & (\theta,\varphi_1) \end{smallmatrix}\right) \in [\tilde{F}\tilde{E}] \cdot [C]$$

  which, after computing the action using Lemma 3.2, gives

$$\big(\varphi_1(e), \varphi(e), E\varphi_1 \circ \tau(\_ \otimes e)\big) \in G_2 \cong [\tilde{F}\tilde{E}]_{12}.$$

For the last term, set $\xi = \_\otimes e + y_2\tau(\_\otimes e)$. Then $E\varphi \circ \xi$ is:

$$\begin{aligned} E\varphi \circ \xi &= \_\otimes \varphi(e) + E\theta \circ y_2\tau(\_\otimes e) + E(y_1\varphi_1) \circ y_2\tau(\_\otimes e) \\ &= \_\otimes \varphi(e) + y_2\tau(\_\otimes e.\theta) + y_1y_2E\varphi_1 \circ \tau(\_\otimes e). \end{aligned}$$

Subtracting $\_\otimes \varphi(e) + y_2\tau\big(\_\otimes (\varphi(e) - y_1\varphi_1(e))\big)$ to isolate $y_1y_2\xi'$, we obtain $y_1y_2E\varphi_1 \circ \tau(\_\otimes e)$ and the last component follows. For the final result observe that $E\varphi_1 \circ \tau(\_\otimes e) = \sigma E(e \otimes \varphi_1)$.

- We have $[\tilde\sigma]_{22} : [\tilde E\tilde F]_{22} \to [\tilde F\tilde E]_{22}$ given by:

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & F\varepsilon E & 0 \\ \eta & y_1 & 0 & 0 & \sigma \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & F\sigma E & 0 \end{pmatrix}$$

using the ordered decompositions from Eq. 5.5 and Prop. 3.15:
– $[\tilde E\tilde F]_{22} \cong G_1G_1 \oplus EF[y] \cong A[y] \oplus FE[y] \oplus FE[y] \oplus FEFE[y] \oplus EF[y]$,
– $[\tilde F\tilde E]_{22} \cong U \cong FE[y]^{\oplus 4} \oplus F^2E^2[y]$.
We compute $[\tilde\sigma]_{22}$ first on G_1G_1, and afterwards on $EF[y]$. We can use the same presentation for $[\tilde\eta(1)]_{22}$ as in the calculations for $[\tilde\sigma]_{21}$. Let $(\theta, \varphi_1) \otimes (\theta', \varphi_1') \in G_1G_1$ be an arbitrary generator. Then the presentation for the product in $[\tilde F] \cdot [\tilde E] \cdot [\tilde E] \cdot [\tilde F]$ is:

$$\begin{aligned} &\sum_{a\in Q} \left(\begin{smallmatrix} 0 & 0 \\ 0 & (f_a,0,0) \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 0 & 0 \\ 0 & (e_a,0,0) \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 0 & 0 \\ (\theta,\varphi_1) & 0 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 0 & (\theta',\varphi_1') \\ 0 & 0 \end{smallmatrix}\right) \\ &+ \sum_{b\in Q} \left(\begin{smallmatrix} 0 & 0 \\ 0 & (0,f_b,0) \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 0 & 0 \\ 0 & (0,e_b,0) \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 0 & 0 \\ (\theta,\varphi_1) & 0 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 0 & (\theta',\varphi_1') \\ 0 & 0 \end{smallmatrix}\right) \\ &\in \left(\begin{smallmatrix} F[y] & L_1 \\ F^2[y] & L_2 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} E[y] & E^2[y] \\ G_1 & G_2 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} E[y] & E^2[y] \\ G_1 & G_2 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} F[y] & L_1 \\ F^2[y] & L_2 \end{smallmatrix}\right). \end{aligned}$$

Using again the calculations for $[\tilde\sigma]_{21}$, we see that application of $[\tilde F\tilde E\tilde\varepsilon] \circ [\tilde F\tilde\tau\tilde F]$ yields:

$$\begin{pmatrix} 0 & 0 \\ 0 & (\varphi_1, 0, \theta, 0, 0) \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & (\theta', \varphi_1') \end{pmatrix} \in [\tilde F\tilde E] \cdot [C].$$

Now compute the action of (θ', φ_1') on the right on U using Lemma 3.2. For the matrix part $[\Phi]$, we have:

$$\begin{pmatrix} \varphi' & 0 \\ \varphi_1' & \theta' \end{pmatrix} \cdot \begin{pmatrix} \varphi_1 & \theta \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \varphi' \circ \varphi_1 & \theta.\varphi' \\ \varphi_1' \circ \varphi_1 & \theta.\varphi_1' \end{pmatrix}.$$

The submodule form of $(\varphi_1, 0, \theta, 0, 0)$ is $(\varphi_1, 0, \theta, 0, \tau \circ E\varphi)$ using:

$$\begin{aligned} \Lambda &= \tau y_1(E\varphi_1 + E\theta \circ \tau) \\ &= \tau \circ E(y_1\varphi_1) + \tau \circ E\theta \\ &= \tau \circ E\varphi. \end{aligned}$$

Then after taking the action, the last coefficient of the submodule form is given by post-composing with $E\varphi'$ to obtain $\Lambda = E\varphi' \circ \tau \circ E\varphi$, which we expand using $\varphi = \_.\theta + y_1\varphi_1$ and similarly for φ':

$$\begin{aligned}\Lambda &= E\varphi' \circ \tau \circ E\varphi \\ &= E\theta' \circ \tau \circ E\theta + E\theta' \circ \tau \circ E(y_1\varphi_1) \\ &\quad + E(y_1\varphi_1') \circ \tau \circ E\theta + E(y_1\varphi_1') \circ \tau \circ E(y_1\varphi_1).\end{aligned}$$

To compute the bimodule form, evaluate Eq. (3.3) using $[\Phi]$:

$$\begin{aligned}&\Lambda = \\ &\tau y_1\big(E(\varphi' \circ \varphi_1) + E(\theta.\varphi') \circ \tau\big) - y_2\tau y_1\big(E(\varphi_1' \circ \varphi_1) + E(\theta.\varphi_1') \circ \tau\big) + y_1y_2\Lambda^\circ \\ &= \tau y_1 \circ E\varphi' \circ \big(E\varphi_1 + E\theta \circ \tau\big) - y_2\tau \circ E(y_1\varphi_1') \circ \big(E\varphi_1 + E\theta \circ \tau\big) + y_1y_2\Lambda^\circ \\ &= \tau y_1 \circ E\theta' \circ \big(E\varphi_1 + E\theta \circ \tau\big) + E(y_1\varphi_1') \circ \big(E\varphi_1 + E\theta \circ \tau\big) + y_1y_2\Lambda^\circ.\end{aligned}$$

(For the last equality we expand $\varphi' = \_.\theta' + y_1\varphi_1'$ and use the relation $\tau y_1 - y_2\tau = \mathrm{Id}$.) By identifying the two expressions we can solve to find $\Lambda^\circ = E\varphi_1' \circ \tau \circ E\varphi_1$. So the image is given using the bimodule form of U by:

$$\begin{pmatrix} 0 & 0 \\ 0 & \big(\varphi' \circ \varphi_1, \varphi_1' \circ \varphi_1, \theta.\varphi', \theta.\varphi_1', E\varphi_1' \circ \tau \circ E\varphi_1\big) \end{pmatrix} \in \begin{pmatrix} G_1 & G_2 \\ L_2 & U \end{pmatrix} = [\tilde{F}\tilde{E}].$$

Using the fact that $E\varphi_1' \circ \tau \circ E\varphi_1 = F\sigma E(\varphi_1 \otimes \varphi_1')$ and $\varphi_1' \circ \varphi_1 = F\varepsilon E(\varphi_1 \otimes \varphi_1')$, one recovers the first four columns of the matrix of $[\tilde{\sigma}]_{22}$.

For the fifth column of $[\tilde{\sigma}]_{22}$, we start with an arbitrary generator $e \otimes f' \in EF[y] \subset [\tilde{E}\tilde{F}]_{22}$. The element $(0, e, 0) \otimes (f', 0, 0) \in (G_2)_G L_2$ is sent by Γ_{22} of $[\tilde{E}\tilde{F}]$ to $e \otimes f'$. So we consider the element:

$$\begin{aligned}&\sum_{a \in Q} \left(\begin{smallmatrix} 0 & 0 \\ 0 & (f_a,0,0) \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 0 & 0 \\ 0 & (e_a,0,0) \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 0 & 0 \\ 0 & (0,e,0) \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 0 & 0 \\ 0 & (f',0,0) \end{smallmatrix}\right) \\ &+ \sum_{b \in Q} \left(\begin{smallmatrix} 0 & 0 \\ 0 & (0,f_b,0) \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 0 & 0 \\ 0 & (0,e_b,0) \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 0 & 0 \\ 0 & (0,e,0) \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 0 & 0 \\ 0 & (f',0,0) \end{smallmatrix}\right) \\ &\in \begin{pmatrix} F[y] & L_1 \\ F^2[y] & L_2 \end{pmatrix} \cdot \begin{pmatrix} E[y] & E^2[y] \\ G_1 & G_2 \end{pmatrix} \cdot \begin{pmatrix} E[y] & E^2[y] \\ G_1 & G_2 \end{pmatrix} \cdot \begin{pmatrix} F[y] & L_1 \\ F^2[y] & L_2 \end{pmatrix},\end{aligned}$$

and we compute its image under $\tilde{F}\tilde{E}\tilde{\varepsilon} \circ \tilde{F}\tilde{\tau}\tilde{F}$. First apply Γ_{22} of $[\tilde{E}\tilde{E}]$ to $(e_a, 0, 0) \otimes (0, e, 0)$ and $(0, e_b, 0) \otimes (0, e, 0)$, using the rule for bimodule forms on p. 23:

$$\begin{aligned}(e_a, 0, 0) \otimes (0, e, 0) &\xmapsto{\Gamma_{22}} (0, -y_2(e_a \otimes e), -y_2(e_a \otimes e), 0) \in G_3, \\ (0, e_b, 0) \otimes (0, e, 0) &\xmapsto{\Gamma_{22}} (\tau y_1(e_b \otimes e), e_b \otimes e, e_b \otimes e, 0) \in G_3.\end{aligned}$$

Next we apply $[\tilde{\tau}]_{22}$ to these elements:

$$\begin{aligned}(0, -y_2(e_a \otimes e), -y_2(e_a \otimes e), 0) &\xmapsto{[\tilde{\tau}]_{22}} \big(e_a \otimes e, e_a \otimes e, -\tau y_2(e_a \otimes e), 0\big), \\ (\tau y_1(e_b \otimes e), e_b \otimes e, e_b \otimes e, 0) &\xmapsto{[\tilde{\tau}]_{22}} \big(\tau(e_b \otimes e), \tau(e_b \otimes e), \tau(e_b \otimes e), 0\big).\end{aligned}$$

Note that formula (2.7) is given for the submodule form of G_3. Using Prop. 3.21 of [McM22], one defines a bimodule form in the usual way, where the last coefficient is χ'' instead of χ. By studying the proof of Lemma 4.3 of [McM22], one observes that the action of $\tilde{\tau}$ on the last coefficient in this bimodule form is (also) given by post-composition with τE, whence the final zeros above.

The next step is to express $\big(e_a e, e_a e, -\tau y_2(e_a e), 0\big)$ and $\big(\tau(e_b e), \tau(e_b e), \tau(e_b e), 0\big)$ back in $(G_2)_G G_2$ (i.e. find a preimage under $\Gamma_{22}|_{(G_2)_G G_2}$) in order to view them in $[\tilde{E}]\cdot[\tilde{E}]$. We will need the notation $\tau(ee) = \sum_{d \in P} \tau(ee)_{(1d)} \otimes \tau(ee)_{(2d)}$ introduced to compute $[\tilde{\sigma}_{21}]$ above.

Claim. We have:

$$\sum_{d\in P} \begin{matrix} \big(0,\tau(e_a e)_{(1d)},0\big)\otimes\big(\tau(e_a e)_{(2d)},y_1\tau(e_a e)_{(2d)},0\big) \\ -\big(0,\tau y_1(e_a e)_{(1d)},0\big)\otimes\big(0,\tau y_1(e_a e)_{(2d)},0\big) \end{matrix} \xmapsto{\Gamma_{22}} \big(e_a e, e_a e, -\tau y_2(e_a e), 0\big),$$

$$\sum_{d\in P} \big(0, \tau(e_b e)_{(1d)}, 0\big) \otimes \big(0, \tau(e_b e)_{(2d)}, 0\big) \xmapsto{\Gamma_{22}} \big(\tau(e_b e), \tau(e_b e), \tau(e_b e), 0\big).$$

Proof. The proof is a direct calculation using the bimodules formulation of $\Gamma_{22}|_{(G_2)_G G_2}$ on p. 23. □

Thus, after applying $\tilde{F}\tilde{\tau}\tilde{F}$, we have the element:

$$\sum_{a\in Q, d\in P} \begin{pmatrix} 0 & 0 \\ 0 & (f_a,0,0) \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & \big(0,\tau(e_a e)_{(1d)},0\big) \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & \big(\tau(e_a e)_{(2d)},y_1\tau(e_a e)_{(2d)},0\big) \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & (f',0,0) \end{pmatrix}$$
$$+ \sum_{a\in Q, d\in P} \begin{pmatrix} 0 & 0 \\ 0 & (f_a,0,0) \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & -\big(0,\tau y_1(e_a e)_{(1d)},0\big) \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & \big(0,\tau y_1(e_a e)_{(2d)},0\big) \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & (f',0,0) \end{pmatrix}$$
$$+ \sum_{b\in Q, d\in P} \begin{pmatrix} 0 & 0 \\ 0 & (0,f_b,0) \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & \big(0,\tau(e_b e)_{(1d)},0\big) \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & \big(0,\tau(e_b e)_{(2d)},0\big) \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & (f',0,0) \end{pmatrix},$$

and we need to apply $[\tilde{F}\tilde{E}] \cdot \tilde{\varepsilon}$ and then realize the result in $[\tilde{F}\tilde{E}]$. Observe that:

$$\big(0, \tau y_1(e_a e)_{(2d)}, 0\big) \otimes (f', 0, 0) \xmapsto{\tilde{\varepsilon}} 0,$$
$$\big(0, \tau(e_b e)_{(2d)}, 0\big) \otimes (f', 0, 0) \xmapsto{\tilde{\varepsilon}} 0.$$

Therefore only the top row will remain. We have in submodule form:

$$\big(\tau(e_a e)_{(2d)}, y_1\tau(e_a e)_{(2d)}, 0\big) \otimes (f', 0, 0)$$
$$\xmapsto{[\tilde{\varepsilon}]_{22}} \Big(f'\big(\tau(e_a e)_{(2d)}\big), Ef' \circ \tau \circ \big(\_ \otimes y_1\tau(e_a e)_{(2d)}\big)\Big) \in G_1.$$

We convert to bimodule form and give this a name:

$$(\theta, \varphi_1)_{a,d} := \Big(f'\big(\tau(e_a e)_{(2d)}\big), Ef' \circ \tau\big(\_ \otimes \tau(e_a e)_{(2d)}\big)\Big) \in G_1.$$

Observe that under the composition isomorphism $(L_2)_G G_2 \xrightarrow{\sim} U$ we have:

$$(f_a, 0, 0) \otimes \big(0, \tau(e_a e)_{(1d)}, 0\big) \mapsto \big(0, 0, f_a(\_).\tau(e_a e)_{(1d)}, 0, 0\big) \in U.$$

We are therefore left with:

$$\sum_{a\in Q, d\in P} \begin{pmatrix} 0 & 0 \\ 0 & \big(0, 0, f_a(\_).\tau(e_a e)_{(1d)}, 0, 0\big) \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & (\theta, \varphi_1)_{a,d} \end{pmatrix} \in [\tilde{F}\tilde{E}] \cdot [C].$$

It remains to use the right action of G_1^{op} on R (Lemma 3.2) to compute the action of $(\theta,\varphi_1)_{a,d}$. The new matrix is given for each term of the sum by:

$$\begin{pmatrix} Ef'\circ\tau y_1\big(\_\otimes\tau(e_ae)_{(2d)}\big) & 0\\ Ef'\circ\tau(\_\otimes\tau(e_ae)_{(2d)}) & f'\big(\tau(e_ae)_{(2d)}\big)\end{pmatrix}\cdot\begin{pmatrix}0 & f_a(\_).\tau(e_ae)_{(1d)}\\ 0 & 0\end{pmatrix}$$
$$=\begin{pmatrix}0 & Ef'\circ\tau y_1\Big(f_a(\_).\tau(e_ae)_{(1d)}\otimes\tau(e_ae)_{(2d)}\Big)\\ 0 & Ef'\circ\tau\Big(f_a(\_).\tau(e_ae)_{(1d)}\otimes\tau(e_ae)_{(2d)}\Big)\end{pmatrix}.$$

After summing over a and d this becomes:

$$\overset{\Sigma_{a,d}}{\rightsquigarrow}\begin{pmatrix}0 & Ef'\circ\tau\circ y_1\tau(\_\otimes e)\\ 0 & Ef'\circ\tau\big(\tau(\_\otimes e)\big)\end{pmatrix}=\begin{pmatrix}0 & Ef'\circ\tau(\_\otimes e)\\ 0 & 0\end{pmatrix}.$$

This matrix gives the first four components of the final element of U. To find the fifth in submodule form, we compute the submodule form of $\big(0,0,f_a(\_).\tau(e_ae)_{(1d)},0,0\big)$ and post-compose with $E\varphi$:

$$\begin{aligned}
&E\varphi\circ\Big(\tau y_1\circ E\big(f_a(_).\tau(e_ae)_{(1d)}\big)\circ\tau\Big)\\
&=E\varphi\circ\Big(\tau y_1\big(Ef_a\circ\tau(_\,_)\otimes\tau(e_ae)_{(1d)}\big)\Big)\\
&=\Big(E^2f'\circ E\tau\circ y_1\big(_\,_\otimes\tau(e_ae)_{(2d)}\big)\Big)\circ\Big(\tau y_1\big(Ef_a\circ\tau(_\,_)\otimes\tau(e_ae)_{(1d)}\big)\Big)\\
&=E^2f'\circ E\tau\circ y_1\big(\tau y_1\big(Ef_a\circ\tau(_\,_)\otimes\tau(e_ae)_{(1d)}\big)\otimes\tau(e_ae)_{(2d)}\big)\\
&=E^2f'\circ E\tau\circ\tau E\circ y_2y_1\big(Ef_a\circ\tau(_\,_)\otimes\tau(e_ae)_{(1d)}\otimes\tau(e_ae)_{(2d)}\big).
\end{aligned}$$

(The last equality is, schematically, $y_1\big((\tau y_1AA)\otimes B\big)=\tau E\circ y_2y_1(AA\otimes B)$.) Summing over d and a we obtain:

$$\overset{\Sigma_{a,d}}{\rightsquigarrow}E^2f'\circ E\tau\circ\tau E\circ y_2y_1\big(E\tau\circ\tau E(\_\,\_\otimes e)\big).$$

Now observe the following calculation in the nil affine Hecke algebra:

$$\begin{aligned}
\tau_1(\tau_2y_2)y_1\tau_1\tau_2&=\tau_1(y_3\tau_2)y_1\tau_1\tau_2+\tau_1y_1\tau_1\tau_2\\
&=(\tau_1y_3)(\tau_2y_1)\tau_1\tau_2+(\tau_1y_1)\tau_1\tau_2\\
&=(y_3\tau_1)(y_1\tau_2)\tau_1\tau_2+(y_2\tau_1)\tau_1\tau_2+\tau_1\tau_2\\
&=y_3(\tau_1y_1)\tau_2\tau_1\tau_2+0+\tau_1\tau_2\\
&=y_3(y_2\tau_1)\tau_2\tau_1\tau_2+y_3\tau_2\tau_1\tau_2+\tau_1\tau_2\\
&=0+y_3\tau_2\tau_1\tau_2+\tau_1\tau_2.
\end{aligned}$$

(Here $\tau_i=E^{n-i-1}\tau E^{i-1}$ for whatever n.) Therefore we have:

$$\begin{aligned}
&E^2f'\circ E\tau\circ\tau E\circ y_2y_1\big(E\tau\circ\tau E(_\,_\otimes e)\big)\\
&=y_2E^2f'\circ\tau E\circ E\tau\circ\tau E(_\,_\otimes e)+E^2f'\circ E\tau\circ\tau E(_\,_\otimes e).
\end{aligned}$$

Now to find the bimodule form of the fifth component we consider:

$$\begin{aligned}
&\tau y_1 \circ \big(E^2 f' \circ E\tau(__ \otimes e) \circ \tau\big) \\
&= \tau y_1 \circ E^2 f' \circ E\tau \circ \tau E(__ \otimes e) \\
&= y_2 E^2 f' \circ \tau E \circ E\tau \circ \tau E(__ \otimes e) + E^2 f' \circ E\tau \circ \tau E(__ \otimes e),
\end{aligned}$$

and since this agrees with the expression before it, Eq. (3.3) implies that the fifth component in bimodule form is zero. The final expression is $\big(0, 0, Ef' \circ \tau\big(\_ \otimes e\big), 0, 0\big) \in U \cong [\tilde{F}\tilde{E}]_{22}$. Observe that $Ef' \circ \tau(\_ \otimes e) = \sigma(e \otimes f')$. This gives the fifth column of the matrix of $[\tilde{\sigma}]_{22}$, and we have now justified all components of that matrix.

5.2.2. *Maps $\tilde{\varepsilon} \circ \tilde{x}^i \tilde{F}$ and $\tilde{F}\tilde{x}^i \circ \tilde{\eta}$.* We continue by computing the maps $\tilde{\varepsilon} \circ \tilde{x}^i \tilde{F}$ and $\tilde{F}\tilde{x}^i \circ \tilde{\eta}$ on the various components of the matrices $[\tilde{E}\tilde{F}]$, $[\tilde{F}\tilde{E}]$, and $[C]$. As before, we propose these maps in the bulleted lines and justify them in the paragraphs following.

- We have $[\tilde{\varepsilon} \circ \tilde{x}^i \tilde{F}]_{11} : [\tilde{E}\tilde{F}]_{11} \to [C]_{11}$ given by $\varepsilon \circ x^i y_1 F$ using the decompositions:
  - $[\tilde{E}\tilde{F}]_{11} \cong EF[y]$,
  - $[C]_{11} \cong A[y]$.

  The endomorphism $\tilde{x} \in \mathrm{End}(\tilde{E})$ (see (2.6)) determines an endomorphism of $[\tilde{E}\tilde{F}]_{11}$ given by xF on $EF[y]$. The morphism $\tilde{\varepsilon}$ composes elements of $\tilde{E}$ with those of $\tilde{F}$ when they are interpreted in $\mathrm{Hom}_{D^b(B)}(X, E'X)$ and $\mathrm{Hom}_{D^b(B)}(E'X, X)$. In particular, $e \in E[y] \cong [\tilde{E}]_{11}$ represents the morphism $X_1 \to E'X_1$ given by $1 \mapsto y_1 e$ in degree 0 of the top row, and $f \in F[y] \cong [\tilde{F}]_{11}$ represents the morphism given by $e \mapsto f(e)$ in degree 0 of the top row.
- We have $[\tilde{F}\tilde{x}^i \circ \tilde{\eta}]_{11} : [C]_{11} \to [\tilde{F}\tilde{E}]_{11}$ given by $\begin{pmatrix} y^i \\ Fh_{i-1}(x,y)\circ\eta \end{pmatrix}$ using the decompositions:
  - $[C]_{11} \cong A[y]$,
  - $[\tilde{F}\tilde{E}]_{11} \cong G_1 \cong A[y] \oplus FE[y]$.

  Here $h_i(z_1, \ldots, z_n)$ is the complete homogeneous symmetric polynomial of degree i in the variables $z_1, \ldots, z_n$. Note the small case interpretations:

$$\begin{cases}
h_{i-1}(x,y) = 0 & i = 0 \\
h_{i-1}(x,y) = 1 & i = 1 \\
h_{i-1}(x,y) = x + y & i = 2 \\
\ldots & \ldots
\end{cases}$$

Observe that $[\tilde{\eta}]_{11}$ is given by $1 \mapsto \mathrm{Id}_{X_2} \in G_1^{\mathsf{op}} \cong \mathrm{End}_{K^b(B)}(X_2)$, and $\mathrm{Id}_{X_2} = (1, 0)$ (in bimodule form). More generally $\theta \mapsto \_.\theta \in \mathrm{Hom}_A({}_AE, E)[y] \cong FE[y] \subset G_1^{\mathsf{op}}$. From (2.6) we have the action of $[\tilde{x}]_{11}$ on G_1^{op} in submodule form: $\tilde{x}^i.(\theta, \varphi) = (y^i\theta, x^i \circ \varphi)$. Now convert this expression to bimodule form:

$$\begin{aligned} x^i \circ \varphi &= x^i \circ \_.\theta + x^i y_1 \varphi_1 \\ &= y^i\theta + (x^i - y^i) \circ \_.\theta + y_1 x^i \varphi_1 \\ &= y^i\theta + y_1\big(h_{i-1}(x,y) \circ \_.\theta + x^i \circ \varphi_1\big), \end{aligned}$$

so $\tilde{x}^i.(\theta,\varphi_1) = (y^i\theta, h_{i-1}(x,y) \circ \_.\theta + x^i \circ \varphi_1)$. In particular, $\tilde{x}^i.(1,0) = (y^i, h_{i-1}(x,y))$, which gives the proposed formula by viewing x, y as endofunctors of E instead of as elements of $FE[y]$.

- We have $[\tilde{\varepsilon} \circ \tilde{x}^i\tilde{F}]_{21} : [\tilde{E}\tilde{F}]_{21} \to [C]_{21}$ given by $\big(x^i, F(\varepsilon \circ x^i y_1 F)\big)$ using the decompositions:
  - $[\tilde{E}\tilde{F}]_{21} \cong G_1F[y] \cong F[y] \oplus FEF[y]$,
  - $[C]_{21} \cong F[y]$.

  (Here $x \in \mathrm{End}(F)[y]$ is given by $x(f) = f \circ x$.) The map $[\tilde{\varepsilon}]_{21} : G_1F[y] \to F[y]$ is given (using submodule form) by $(\theta,\varphi) \otimes f \mapsto f \circ \varphi$. The endomorphism $[\tilde{x}]_{21}$ acts on G_1 as described under the previous bullet: $\tilde{x}^i.(\theta,\varphi_1) = (y^i\theta, h_{i-1}(x,y) \circ \_.\theta + x^i \circ \varphi_1)$. Then $[\tilde{\varepsilon} \circ \tilde{x}^i\tilde{F}]_{21} : G_1F[y] \to F[y]$ is given using bimodule form by:

$$\begin{aligned} \tilde{x}^i.(\theta,\varphi_1) \otimes f &\mapsto f \circ x^i \circ \varphi \\ &\qquad = f \circ x^i \circ \_.\theta + f \circ x^i y_1 \varphi_1, \end{aligned}$$

  and the component data follows from this formula.

- We have $[\tilde{F}\tilde{x}^i \circ \tilde{\eta}]_{21} : [C]_{21} \to [\tilde{F}\tilde{E}]_{21}$ given by $\begin{pmatrix} 0 \\ y^i \\ F(Fh_{i-1}(x,y)\circ\eta) \end{pmatrix}$ using the decompositions:
  - $[C]_{21} \cong F[y]$,
  - $[\tilde{F}\tilde{E}]_{21} \cong L_2 \cong F[y] \oplus F[y] \oplus F^2E[y]$.

  Let $\left(\begin{smallmatrix} 0 & 0 \\ f & 0 \end{smallmatrix}\right) \in \begin{pmatrix} A[y] & E[y] \\ F[y] & G_1^{\mathrm{op}} \end{pmatrix} = [C]$, and observe that:

$$\begin{aligned} \tilde{\eta}\left(\left(\begin{smallmatrix} 0 & 0 \\ f & 0 \end{smallmatrix}\right)\right) &= \tilde{\eta}\left(\left(\begin{smallmatrix} 0 & 0 \\ f & 0 \end{smallmatrix}\right) . \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} 0 & 0 \\ f & 0 \end{smallmatrix}\right) .\tilde{\eta}\left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\right) \\ &= \left(\begin{smallmatrix} 0 & 0 \\ f & 0 \end{smallmatrix}\right) . \left(\begin{smallmatrix} (1,0) & 0 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ (0,f,0) & 0 \end{smallmatrix}\right) \in \left(\begin{smallmatrix} G_1 & G_2 \\ L_2 & U \end{smallmatrix}\right) = [\tilde{F}\tilde{E}]. \end{aligned}$$

  Here $(0,f,0)$ is written in the bimodule form of L_2. (The action of $f \in F[y] \subset [C]_{21}$ on generators in $G_1 \subset [\tilde{F}\tilde{E}]$ is given by $F[y]G_1 \to L_2$, $f \otimes (\theta,\varphi) \mapsto (0, f \circ \_.\theta, \varphi \circ Ef)$ (written in submodule form), and this image is $(0, f \circ \_.\theta, \varphi_1 \circ Ef)$ in bimodule form.)

  Now we apply $[\tilde{F}\tilde{x}^i]_{21}$. From Eq. (5.3):

$$[\tilde{F}] \cdot [\tilde{E}] \supset (L_2)_G G_1 \ni (0,f,0) \otimes (1,0) \xmapsto{\Gamma_{21}} (0,f,0) \in L_2 \subset [\tilde{F}\tilde{E}].$$

  We have already seen that $\tilde{x}^i.(1,0) = (y^i, h_{i-1}(x,y)) \in G_1$, so we have:

$$(0,f,0) \otimes (1,0) \xmapsto{[\tilde{F}\tilde{x}^i]_{21}} (0,f,0) \otimes (y^i, h_{i-1}(x,y)).$$

  Then

$$\Gamma_{21} : (0,f,0) \otimes (y^i, h_{i-1}(x,y)) \mapsto \big(0, y^i f, x^i \circ Ef\big)$$

  written in submodule form. In bimodule form the image is:

$$\big(0, y^i f, h_{i-1}(x,y) \circ Ef\big),$$

which we compute using:

$$\begin{aligned} x^i \circ Ef &= \big(y^i + y_1 h_{i-1}(x,y)\big) \circ Ef \\ &= E(y^i f) + y_1\big(h_{i-1}(x,y) \circ Ef\big). \end{aligned}$$

Note that $F^2E[y] \ni h_{i-1}(x,y) \circ Ef = F\big(h_{i-1}(x,y) \circ \eta\big)(f)$.

- We have $[\tilde{\varepsilon} \circ \tilde{x}^i \tilde{F}]_{12} : [\tilde{E}\tilde{F}]_{12} \to [C]_{12}$ given by $\big(x^i, (\varepsilon \circ x^i y_1 F)E\big)$ using the decompositions:
  - $[\tilde{E}\tilde{F}]_{12} \cong E[y]G_1 \cong E[y] \oplus EFE[y]$,
  - $[C]_{12} \cong E[y]$.

  The endomorphism $[\tilde{x}]_{12}$ acts as x on $E[y] = [\tilde{E}]_{11}$, and thus as xG_1 on $E[y]G_1 = [\tilde{E}\tilde{F}]_{12}$. The map $[\tilde{\varepsilon}]_{12} : E[y]G_1 \to E[y]$ is given (using submodule form) by $e \otimes (\theta, \varphi) \mapsto y_1^{-1}\varphi(y_1 e)$. (Recall that $e \in E[y]$ indicates the map $X_1 \to X_2$ given on the top row by $A[y] \to E[y]$, $1 \mapsto y_1 e$.) So we have:

$$x^i(e) \otimes (\theta, \varphi_1) \xmapsto{[\tilde{\varepsilon}]_{12}} y_1^{-1}\varphi(x^i y_1 e) = x^i(e).\theta + \varphi_1(x^i y_1 e),$$

  and the component data follows from this formula.
- We have $[\tilde{F}\tilde{x}^i \circ \tilde{\eta}]_{12} : [C]_{12} \to [\tilde{F}\tilde{E}]_{12}$ given by $\begin{pmatrix} y^i \\ y^i y_1 \\ (Fh_{i-1}(x,y)\circ\eta)E \end{pmatrix}$ using the decompositions:
  - $[C]_{12} \cong E[y]$,
  - $[\tilde{F}\tilde{E}]_{12} \cong G_2 \cong E[y] \oplus E[y] \oplus FE^2[y]$.

  By reasoning as in the $[\tilde{F}\tilde{x}^i \circ \tilde{\eta}]_{21}$ case, we find:

$$[C] \ni \left(\begin{smallmatrix} 0 & e \\ 0 & 0 \end{smallmatrix}\right) \xmapsto{[\tilde{\eta}]} \left(\begin{smallmatrix} 0 & (e, y_1 e, 0) \\ 0 & 0 \end{smallmatrix}\right) \in \left(\begin{smallmatrix} G_1 & G_2 \\ L_2 & U \end{smallmatrix}\right) = [\tilde{F}\tilde{E}],$$

  using the bimodule form of G_2. Now we apply $[\tilde{F}\tilde{x}^i]_{12}$. From Eq. (5.3):

$$[\tilde{F}] \cdot [\tilde{E}] \supset (L_1)_G G_2 \ni (1,0) \otimes (e, y_1 e, 0) \xmapsto{\Gamma_{12}} (e, y_1 e, 0) \in G_2 \subset [\tilde{F}\tilde{E}].$$

  In (2.6) we have a formula for the action of $[\tilde{x}^i]_{22}$ on $G_2 \subset [\tilde{E}]$ written in terms of the data e_1, e_2, ξ. The data $(e, y_1 e, 0)$ corresponds to $e_1 = y_1 e$, $e_2 = 0$, $\xi = {}_{-} \otimes y_1 e$ (see the paragraph after Prop. 3.8). Applying $[\tilde{x}^i]_{22}$ gives $e_1 = y^i y_1 e$, $e_2 = 0$, $\xi = {}_{-} \otimes y^i y_1 e + y_1 y_2 h_{i-1}(x_2, y)({}_{-} \otimes e)$, where to compute ξ we have used:

$$\begin{aligned} x_2^i \circ ({}_{-} \otimes y_1 e) &= \big(y^i + y_2 h_{i-1}(x_2, y)\big) \circ ({}_{-} \otimes y_1 e) \\ &= {}_{-} \otimes y^i y_1 e + y_1 y_2 h_{i-1}(x_2, y)({}_{-} \otimes e). \end{aligned}$$

  This corresponds to the data $\big(y^i e, y^i y_1 e, h_{i-1}(x_2, y)({}_{-} \otimes e)\big) \in G_2$ in the bimodule form. So we have:

$$\begin{aligned} (1,0) \otimes (e, y_1 e, 0) &\xmapsto{[\tilde{F}\tilde{x}^i]_{12}} (1,0) \otimes \big(y^i e, y^i y_1 e, h_{i-1}(x_2, y)({}_{-} \otimes e)\big) \\ &\xmapsto{\Gamma_{12}} \big(y^i e, y^i y_1 e, h_{i-1}(x_2, y)({}_{-} \otimes e)\big) \in G_2 \subset [\tilde{F}\tilde{E}]. \end{aligned}$$

  Note that $FE^2[y] \ni h_{i-1}(x_2, y)({}_{-} \otimes e) = \big((Fh_{i-1}(x,y) \circ \eta)E\big)(e)$.
- We have $[\tilde{\varepsilon} \circ \tilde{x}^i \tilde{F}]_{22} : [\tilde{E}\tilde{F}]_{22} \to [C]_{22}$ given by:

$$\begin{pmatrix} y^i & 0 & 0 & 0 & -\varepsilon \circ h_{i-1}(x,y)F \\ h_{i-1}(x,y) \circ \eta & x^i E & Fx^i & F(\varepsilon \circ x^i y_1 F)E & \substack{-FE\varepsilon\circ F(\tau\circ h_{i-1}(x_1,x_2))F\circ\eta EF \\ -FE\varepsilon\circ F(h_{i-2}(x_1,x_2,y))F\circ\eta EF} \end{pmatrix}$$

using the ordered decompositions (recall Eq. (5.5)):
– $[\tilde{E}\tilde{F}]_{22} \cong G_1G_1 \oplus EF[y] \cong A[y] \oplus FE[y] \oplus FE[y] \oplus FEFE[y] \oplus EF[y]$,
– $[C]_{22} \cong G_1 \cong A[y] \oplus FE[y]$.
Consider the first four columns first, i.e. the restriction of the map to G_1G_1. Take an arbitrary generator $(\theta, \varphi_1) \otimes (\theta', \varphi_1')$. Borrowing a calculation from the case $[\tilde{\varepsilon} \circ \tilde{x}^i\tilde{F}]_{21}$ we find:

$$(\theta, \varphi_1) \otimes (\theta', \varphi_1') \overset{[\tilde{x}^i\tilde{F}]_{22}}{\longmapsto} (y^i\theta, h_{i-1}(x,y) \circ \_.\theta + x^i \circ \varphi_1) \otimes (\theta', \varphi_1').$$

Now $[\tilde{\varepsilon}]_{22} : G_1G_1 \to G_1$ is given by composition, so we have:

$$\begin{aligned}
&(y^i\theta, h_{i-1}(x,y) \circ _.\theta + x^i \circ \varphi_1) \otimes (\theta', \varphi_1') \\
&\quad \overset{[\tilde{\varepsilon}]_{22}}{\longmapsto} \big(y^i\theta\theta', _.\theta' \circ h_{i-1}(x,y) \circ _.\theta + (_.\theta') \circ x^i \circ \varphi_1 + \varphi_1' \circ (_.y^i\theta) \\
&\qquad\qquad + \varphi_1' \circ (x^i - y^i) \circ _.\theta + \varphi_1' \circ y_1x^i \circ \varphi_1\big) \\
&\quad = \big(y^i\theta\theta', h_{i-1}(x,y) \circ _.\theta\theta' + \varphi_1' \circ x^i \circ _.\theta + x^i \circ _.\theta' \circ \varphi_1 + \varphi_1' \circ y_1x^i \circ \varphi_1\big).
\end{aligned}$$

The first four columns of the matrix of $[\tilde{\varepsilon} \circ \tilde{x}^i\tilde{F}]_{22}$ can be read off this formula.

The last column gives the restriction of $[\tilde{\varepsilon} \circ \tilde{x}^i\tilde{F}]_{22}$ to a map $EF[y] \to A[y] \oplus FE[y]$. Its computation is more involved. We start with a generator $e \otimes f$, and note that:

$$[\tilde{E}] \cdot [\tilde{F}] \supset (G_2)_G L_2 \ni (0, e, 0) \otimes (f, 0, 0) \overset{\Gamma_{22}}{\longmapsto} e \otimes f \in EF[y] \subset [\tilde{E}\tilde{F}]_{22}$$

using $\Gamma_{22} \mid_{(G_2)_G L_2} = \kappa$ from Eq. (5.4). Now we must apply $[\tilde{x}]_{22}$ to the first factor, and then compose the factors, thereby applying $[\tilde{\varepsilon}]_{22}$ and giving an element of $G_2 \cong \mathrm{End}_{K^b(B)}(X_2)$.

The data $(0, e, 0)$ corresponds to $e_1 = e_2 = e$, $\xi = \tau y_1(\_ \otimes e)$ (see the paragraph after Prop. 3.8). The action of $[\tilde{x}^i]_{22}$ on $G_2 \subset [\tilde{E}]$ then gives $e_1 = y^ie$, $e_2 = x^ie$, $\xi = x_2^i \circ \tau y_1(\_ \otimes e)$. We can compute the composite with $(f, 0, 0)$ directly using this information. It is given in submodule form by:

$$\begin{aligned}
&\big(f \circ y_1^{-1}(y^ie - x^ie), Ef \circ \tau \circ x_2^i \circ \tau y_1(_ \otimes e)\big) \\
&= \big(f(-h_{i-1}(x,y)e), Ef \circ \tau \circ x_2^i \circ \tau y_1(_ \otimes e)\big) \in G_1.
\end{aligned}$$

It remains to convert this to bimodule form. In the calculation we will use three facts, easily checked by the reader:

– $x_2^i \circ \tau = \tau \circ x_1^i - h_{i-1}(x_1, x_2)$,
– $x_2^j = y^j + y_2h_{i-1}(x_2, y)$,
– $\sum_{j+k=i-1} x_1^j h_{k-1}(x_2, y) = h_{i-2}(x_1, x_2, y)$.

Then we have for the main calculation:

$$\begin{aligned}
&Ef\circ\tau\circ x_2^i\circ\tau y_1(_\otimes e)\\
&=-Ef\circ\tau y_1\circ h_{i-1}(x_1,x_2)(_\otimes e)\\
&=-Ef\circ h_{i-1}(x_1,x_2)(_\otimes e)-y_1Ef\circ\tau\circ h_{i-1}(x_1,x_2)(_\otimes e)\\
&=-Ef\circ\sum_{j+k=i-1}x_1^j\big(y^k+y_2h_{k-1}(x_2,y)\big)(_\otimes e)-y_1\circ Ef\circ\tau\circ h_{i-1}(x_1,x_2)(_\otimes e)\\
&=-Ef\circ h_{i-1}(x_1,y)(_\otimes e)-y_1Ef\circ\Big(h_{i-2}(x_1,x_2,y)(_\otimes e)+\tau\circ h_{i-1}(x_1,x_2)(_\otimes e)\Big).
\end{aligned}$$

Then observe that:

$$\begin{aligned}
-Ef\circ h_{i-1}(x_1,y)(_\otimes e)&=_\otimes f(-h_{i-1}(x,y)e)\\
&=(-\varepsilon\circ h_{i-1}(x,y)F)(e\otimes f),
\end{aligned}$$

and that:

$$\begin{aligned}
&-Ef\circ\Big(h_{i-2}(x_1,x_2,y)(_\otimes e)+\tau\circ h_{i-1}(x_1,x_2)(_\otimes e)\Big)\\
&=\Big(-Ef\circ F\big(\tau\circ h_{i-1}(x_1,x_2)+h_{i-2}(x_1,x_2,y)\big)\circ\eta E\Big)(e)\\
&=\Big(-FE\varepsilon\circ F\big(\tau\circ h_{i-1}(x_1,x_2)+h_{i-2}(x_1,x_2,y)\big)F\circ\eta EF\Big)(e\otimes f).
\end{aligned}$$

(We are using that $\_\otimes e$ considered in $\mathrm{Hom}_A({}_AE,E^2)[y]$ corresponds to $(\eta E)(e)$ in $FE^2[y]$; also note that $\varepsilon(e\otimes f)=f(e)\in A[y]$ induces $Ef(\_\otimes e)=\_.f(e)$ considered in $\mathrm{Hom}_A({}_AE,E)[y]$.) The formulas in the last column of $[\tilde\varepsilon\circ\tilde x^i\tilde F]_{22}$ follow.

- We have $[\tilde F\tilde x^i\circ\tilde\eta]_{22}:[C]_{22}\to[\tilde F\tilde E]_{22}$ given by:

$$\begin{pmatrix}
Fy^i\circ\eta & y^iy_1\\
-Fh_{i-1}(x,y)\circ\eta & y^i\\
0 & 0\\
Fx^i\circ\eta & 0\\
F^2\big(h_{i-1}(x_1,x_2)\circ\tau-h_{i-2}(x_1,x_2,y)\big)\circ\eta^2 & F^2h_{i-1}(x_2,y)\circ F\eta E
\end{pmatrix}$$

  using the ordered decompositions:
  – $[C]_{22}\cong G_1\cong A[y]\oplus FE[y]$,
  – $[\tilde F\tilde E]_{22}\cong U\cong FE[y]^{\oplus4}\oplus F^2E^2[y]$.
  Observe first that $[\tilde\eta]_{22}:G_1\to U$ is determined by $(1,0)\mapsto\mathrm{Id}_R=(1,0,0,1,0)\in U$ (using bimodule forms). Recall (Lemma 5.6 used for $[\tilde\sigma]_{21}$) that:

$$\begin{aligned}
(L_2)_GG_2\ni[\tilde\eta(1)]=\sum_{a\in Q}(f_a,0,0)\otimes(e_a,0,0)+\sum_{b\in Q}(0,f_b,0)\otimes(0,e_b,0)\\
\xmapsto{\Gamma_{21}}(1,0,0,1,0)\in U.
\end{aligned}$$

  The map $\Gamma_{21}|_{(L_2)_GG_2}$ of Eq. (5.3) is given by composition and hence right G_1^{op}-equivariant, so we can compute any $[\tilde\eta]_{22}\big((\theta,\varphi_1)\big)$ as $[\tilde\eta(1)].(\theta,\varphi_1)\in(L_2)_GG_2$. The action of $[\tilde F\tilde x^i]$ is applied to elements of $(L_2)_GG_2$, and after that we pass through Γ_{21} again to obtain the final image in U.

We treat the first column of $[\tilde{F}\tilde{x}^i \circ \tilde{\eta}]_{22}$ first, and consider the second column afterwards. For the first column it is enough to consider the case $(\theta, \varphi_1) = (1, 0)$. Starting with the first term, the data $(e_a, 0, 0)$ corresponds to $e_1 = 0$, $e_2 = -y_1 e_a$, and $\xi = y_2\tau(\_\otimes(-y_1 e_a))$. Application of the formula for $[\tilde{x}^i]_{22}$ gives $e_1 = 0$, $e_2 = -x^i y_1 e_a$, and $\xi = x_2^i \circ y_2\tau(\_\otimes(-y_1 e_a))$. Then we convert this to bimodule form, using:

$$\begin{aligned}
&x_2^i \circ y_2\tau(_\otimes(-y_1 e_a)) \\
&= y_2 \circ x_2^i\tau(_\otimes(-y_1 e_a)) \\
&= y_2 \circ \tau x_1^i(_\otimes(-y_1 e_a)) + y_1 y_2 h_{i-1}(x_1, x_2)(_\otimes e_a) \\
&= y_2\tau(_\otimes(-y_1 x^i e_a)) + y_1 y_2 h_{i-1}(x_1, x_2)(_\otimes e_a),
\end{aligned}$$

where in the third line we have used the first fact given under the previous bullet. So in bimodule form we have:

$$[\tilde{x}^i]_{22} : (e_a, 0, 0) \mapsto \big(x^i e_a, 0, h_{i-1}(x_1, x_2)(\_\otimes e_a)\big).$$

Now applying Γ_{21} we obtain:

$$\sum_{a\in Q}\big(x^i e_a, 0, h_{i-1}(x_1, x_2)(\_\otimes e_a)\big) \circ (f_a, 0, 0) = \big(0, 0, 0, x^i, h_{i-1}(x_1, x_2) \circ \tau\big) \in U,$$

where the last component is computed using:

$$\begin{aligned}
&\Big(y_2\tau(_\otimes(-y_1 x^i e_a)) + y_1 y_2 h_{i-1}(x_1, x_2)(_\otimes e_a)\Big) \circ E f_a \circ \tau \\
&= -y_2\tau y_1 x_1^i \tau + y_1 y_2 h_{i-1}(x_1, x_2)\tau,
\end{aligned}$$

together with the facts that $\Phi_{11} = \Phi_{12} = \Phi_{21} = 0$ and $\Phi_{22} = x^i$ so:

$$\begin{aligned}
\Lambda &= \tau y_1(0 + 0) - y_2\tau y_1 \circ (0 + E\Phi_{22} \circ \tau) + y_1 y_2 \Lambda^\circ \\
&= -y_2\tau y_1 \circ x_1^i \circ \tau + y_1 y_2 \Lambda^\circ.
\end{aligned}$$

Continuing with the second term, the data $(0, e_b, 0)$ corresponds to $e_1 = e_b$, $e_2 = e_b$, and $\xi = \tau y_1(\_\otimes e_b)$. Application of the formula for $[\tilde{x}^i]_{22}$ gives $e_1 = y^i e_b$, $e_2 = x^i e_b$, and $\xi = x_2^i \circ \tau y_1(\_\otimes e_b)$. Then we convert this to bimodule form, using:

$$\begin{aligned}
&x_2^i \circ \tau y_1(_\otimes e_b) \\
&= \tau y_1(_\otimes x^i e_b) - y_1 h_{i-1}(x_1, x_2)(_\otimes e_b) \\
&= \tau y_1(_\otimes x^i e_b) - y_1 h_{i-1}(x_1, y)(_\otimes e_b) \\
&\qquad - y_1 y_2 h_{i-2}(x_1, x_2, y)(_\otimes e_b) \\
&= _\otimes x^i e_b + y_2\tau(_\otimes x^i e_b) - _\otimes (x^i - y^i) e_b \\
&\qquad - y_1 y_2 h_{i-2}(x_1, x_2, y)(_\otimes e_b) \\
&= _\otimes y^i e_b + y_2\tau(_\otimes x^i e_b) - y_1 y_2 h_{i-2}(x_1, x_2, y)(_\otimes e_b),
\end{aligned}$$

where we have made use of the fact, easily checked by the reader, that:
$-\, y_2 h_{i-2}(x_1, x_2, y) = h_{i-1}(x_1, x_2) - h_{i-1}(x_1, y)$.

So in bimodule form we have:

$$[\tilde{x}^i]_{22} : (0, e_b, 0) \mapsto \Big(-h_{i-1}(x_1, y)e_b, y^i e_b, -h_{i-2}(x_1, x_2, y)(\_ \otimes e_b)\Big).$$

Now applying Γ_{21} we obtain:

$$\sum_{b \in Q} \Big(-h_{i-1}(x_1, y)e_b, y^i e_b, -h_{i-2}(x_1, x_2, y)(\_ \otimes e_b)\Big) \circ (0, f_b, 0)$$
$$= \big(y^i, -h_{i-1}(x_1, y), 0, 0, -h_{i-2}(x_1, x_2, y)\big) \in U,$$

where the last component is computed using:

$$x_2^i \circ \tau y_1(\_ \otimes e_b) \circ E f_b = x_2^i \tau y_1 = \tau x_1^i y_1 - y_1 h_{i-1}(x_1, x_2)$$

together with the facts that $\Phi_{11} = y^i$, $\Phi_{21} = -h_{i-1}(x_1, y)$, $\Phi_{12} = \Phi_{22} = 0$, so:

$$\begin{aligned} \Lambda &= \tau y_1 \circ \big(y^i + 0 \circ \tau\big) - y_2 \tau y_1 \circ \big(-h_{i-1}(x_1, y) + 0 \circ \tau\big) + y_1 y_2 \Lambda^\circ \\ &= \tau y_1 y^i + y_2 \tau y_1 h_{i-1}(x_1, y) + y_1 y_2 \Lambda^\circ \\ &= \tau y_1 y^i + y_2 \tau (x_1^i - y^i) + y_1 y_2 \Lambda^\circ \\ &= y^i + y_2 \tau x_1^i + y_1 y_2 \Lambda^\circ \\ &= \tau x_1^i y_1 - y_1 h_{i-1}(x_1, y) + y_1 y_2 \Lambda^\circ, \end{aligned}$$

so, using again the fact above:

$$\begin{aligned} y_2 \Lambda^\circ &= -h_{i-1}(x_1, x_2) + h_{i-1}(x_1, y), \\ \Lambda^\circ &= -h_{i-2}(x_1, x_2, y). \end{aligned}$$

Finally taking the sum of the two terms, we conclude that $[\tilde{F}\tilde{x}^i \circ \tilde{\eta}]_{22} : A[y] \to U$ is determined by:

$$1 \mapsto \big(y^i, -h_{i-1}(x_1, y), 0, x^i, h_{i-1}(x_1, x_2) \circ \tau - h_{i-2}(x_1, x_2, y)\big).$$

By describing these coefficients in $FE[y]$ and $F^2E^2[y]$ instead of in $\mathrm{End}(E[y])$ and $\mathrm{End}(E^2[y])$, we obtain the formulas in the first column of the matrix of $[\tilde{F}\tilde{x}^i \circ \tilde{\eta}]_{22}$.

Now we consider the second column of $[\tilde{F}\tilde{x}^i \circ \tilde{\eta}]_{22}$, a map $FE[y] \to U$. It is found using the same method but with $(\theta, \varphi_1) = (0, \varphi_1)$ for a generator $\varphi_1 \in FE[y]$. We have in bimodule form:

$$\begin{aligned} (e_a, 0, 0).(0, \varphi_1) &= \big(0, 0, E\varphi_1 \circ \tau(\_ \otimes -y_1 e_a)\big) \\ (0, e_b, 0).(0, \varphi_1) &= \big(\varphi_1(e_b), y_1 \varphi_1(e_b), E\varphi_1 \circ \tau(\_ \otimes e_b)\big), \end{aligned}$$

where we have used the calculations:

$$E(y_1 \varphi_1) \circ y_2 \tau(\_ \otimes -y_1 e_a) = y_1 y_2 E\varphi_1 \circ \tau(\_ \otimes -y_1 e_a)$$

and

$$E(y_1 \varphi_1) \circ (\_ \otimes e_b + y_2 \tau(\_ \otimes e_b)) = \_ \otimes y_1 \varphi_1(e_b) + y_1 y_2 E\varphi_1 \circ \tau(\_ \otimes e_b).$$

Starting with the first term, the data $\big(0, 0, E\varphi_1 \circ \tau(\_ \otimes -y_1 e_a)\big)$ corresponds to $e_1 = e_2 = 0$ and $\xi = y_1 y_2 E\varphi_1 \circ \tau(\_ \otimes -y_1 e_a)$. Application of the formula for $[\tilde{x}^i]_{22}$ gives $e_1 = e_2 = 0$ and $\xi = x_2^i \circ y_1 y_2 E\varphi_1 \circ \tau(\_\otimes - y_1 e_a)$. Converting this data to bimodule form is trivial. So we have:

$$[\tilde{x}^i]_{22} : \big(0, 0, E\varphi_1 \circ \tau(\_\otimes - y_1 e_a)\big) \mapsto \big(0, 0, x_2^i \circ E\varphi_1 \circ \tau(\_\otimes - y_1 e_a)\big).$$

Now applying Γ_{21} we obtain:

$$\sum_{a \in Q} \big(0, 0, x_2^i \circ E\varphi_1 \circ \tau(\_\otimes - y_1 e_a)\big) \circ (f_a, 0, 0) = \big(0, 0, 0, 0, -E\varphi_1 \circ x_2^i \tau\big) \in U,$$

where the last component is computed using:

$$\begin{aligned} & y_1 y_2 x_2^i \circ E\varphi_1 \circ \tau(\_\otimes - y_1 e_a) \circ E f_a \circ \tau \\ &= -x_2^i \circ y_1 y_2 E\varphi_1 \circ \tau y_1 \circ \tau \\ &= -y_1 y_2 E\varphi_1 \circ x_2^i \tau. \end{aligned}$$

Continuing with the second term, the data $\big(\varphi_1(e_b), y_1\varphi_1(e_b), E\varphi_1 \circ \tau(\_\otimes e_b)\big)$ corresponds to $e_1 = y_1\varphi_1(e_b)$, $e_2 = 0$, and $\xi = \_\otimes y_1\varphi_1(e_b) + y_1 y_2 E\varphi_1 \circ \tau(\_\otimes e_b)$. Application of the formula for $[\tilde{x}^i]_{22}$ gives $e_1 = y_1 y^i \varphi_1(e_b)$, $e_2 = 0$, and $\xi = x_2^i \circ \big(\_\otimes y_1\varphi_1(e_b) + y_1 y_2 E\varphi_1 \circ \tau(\_\otimes e_b)\big)$. Then we convert this to bimodule form, using:

$$\begin{aligned} & x_2^i \circ \big(\_\otimes y_1\varphi_1(e_b) + y_1 y_2 E\varphi_1 \circ \tau(\_\otimes e_b)\big) \\ &= \_\otimes y^i y_1 \varphi_1(e_b) + y_2 h_{i-1}(x_2, y)(\_\otimes y_1\varphi_1(e_b)) + y_1 y_2 E\varphi_1 \circ x_2^i \tau(\_\otimes e_b) \\ &= \_\otimes y^i y_1 \varphi_1(e_b) + y_1 y_2 \Big(h_{i-1}(x_2, y) \circ E\varphi_1(\_\otimes e_b) + E\varphi_1 \circ x_2^i \tau(\_\otimes e_b)\Big) \\ &= \_\otimes y^i y_1 \varphi_1(e_b) + y_1 y_2 E\varphi_1 \circ \big(x_2^i \tau + h_{i-1}(x_2, y)\big)(\_\otimes e_b) \\ &= \_\otimes y_1 y^i \varphi_1(e_b) + y_1 y_2 \Big(-E\varphi_1 \circ y_1 h_{i-2}(x_1, x_2, y)(\_\otimes e_b) + E\varphi_1 \circ \tau \circ x_1^i(\_\otimes e_b)\Big). \end{aligned}$$

So in bimodule form we have:

$$\begin{aligned} & [\tilde{x}^i]_{22} : \big(\varphi_1(e_b), y_1\varphi_1(e_b), E\varphi_1 \circ \tau(\_\otimes e_b)\big) \mapsto \\ & \Big(y^i \varphi_1(e_b), y_1 y^i \varphi_1(e_b), E\varphi_1 \circ \big(x_2^i \tau + h_{i-1}(x_2, y)\big)(\_\otimes e_b)\Big). \end{aligned}$$

Now applying Γ_{21} we obtain:

$$\begin{aligned} & \sum_{b \in Q} \Big(y^i \varphi_1(e_b), y_1 y^i \varphi_1(e_b), E\varphi_1 \circ \big(x_2^i \tau + h_{i-1}(x_2, y)\big)(\_\otimes e_b)\Big) \circ (0, f_b, 0) \\ &= \Big(y^i y_1 \varphi_1, y^i \varphi_1, 0, 0, E\varphi_1 \circ \big(x_2^i \tau + h_{i-1}(x_2, y)\big)\Big) \in U, \end{aligned}$$

where the last component is computed using:

$$\begin{aligned} & \Big(\_\otimes y^i y_1 \varphi_1(e_b) + y_1 y_2 E\varphi_1 \circ \big(x_2^i \tau + h_{i-1}(x_2, y)\big)(\_\otimes e_b)\Big) \circ E f_b \\ &= y^i y_1 E\varphi_1 + y_1 y_2 E\varphi_1 \circ \big(x_2^i \tau + h_{i-1}(x_2, y)\big)(\_\otimes e_b), \end{aligned}$$

together with the facts that $\Phi_{11} = y^i y_1 \varphi_1$, $\Phi_{21} = y^i \varphi_1$, $\Phi_{12} = \Phi_{22} = 0$, so:

$$\begin{aligned} \Lambda &= \tau y_1 (y^i y_1 E\varphi_1 + 0 \circ \tau) - y_2 \tau y_1 (y^i E\varphi_1 + 0 \circ \tau) + y_1 y_2 \Lambda^\circ \\ &= y^i y_1 E\varphi_1 + y_1 y_2 \Lambda^\circ. \end{aligned}$$

Taking the sum of the two terms, we conclude that $[\tilde{F}\tilde{x}^i \circ \tilde{\eta}]_{22} : FE[y] \to U$ is given by:

$$\varphi_1 \mapsto \big(y^i y_1 \varphi_1, y^i \varphi_1, 0, 0, E\varphi_1 \circ h_{i-1}(x_2, y)\big).$$

The last component, an element of $\mathrm{End}_A({}_AE^2)[y]$, is the same as $\big(F^2 h_{i-1}(x_2, y) \circ F\eta E\big)(\varphi_1)$. This gives the formulas in the second column of the matrix of $[\tilde{F}\tilde{x}^i \circ \tilde{\eta}]_{22}$.

5.3. **Maps $\tilde{\rho}_\lambda$: isomorphisms.** Now we have formulas by components for the maps $\tilde{\sigma}$, $\tilde{\varepsilon} \circ \tilde{x}^i\tilde{F}$, and $\tilde{F}\tilde{x}^i \circ \tilde{\eta}$ that are used to define the maps $\tilde{\rho}_\lambda$. It remains to make use of the isomorphisms ρ_λ determined by σ, $\varepsilon \circ x^iF$, and $Fx^i \circ \eta$, together with these formulas, to show that $\tilde{\rho}_\lambda$ are isomorphisms. Note that $\tilde{\rho}_\lambda$ are already known to give morphisms of (C, C)-bimodules, so it suffices to show that $\tilde{\rho}_\lambda$ are isomorphisms of sets. We will work again by components and show that $[\tilde{\rho}_\lambda]_{ij}$ is an isomorphism of $(A[y], A[y])$-bimodules for $i, j \in \{1, 2\}$.

We remind the reader of our notational convention that $E_\lambda = Ee_\lambda$ for the idempotents $e_\lambda \in A_\lambda$ of a weight decomposition. Recall that the bimodule E satisfies $e_jEe_i = \delta_{i+2,j} \cdot e_{i+2}Ee_i$, and similarly for F but with $i-2$ instead of $i+2$. Finally, recall Prop. 4.26 of [McM22] that gives the weight idempotents for the algebra C.

- We have for $[\tilde{\rho}_\lambda]_{11}$, $\lambda \geqslant 0$:

$$[\tilde{\rho}_\lambda]_{11} : EF_{\lambda+1}[y] \to A_{\lambda+1}[y] \oplus FE_{\lambda+1}[y] \oplus A_{\lambda+1}[y]^{\oplus\lambda}$$

given by:

$$[\tilde{\rho}_\lambda]_{11} = \varepsilon \oplus \sigma \oplus \bigoplus_{i=0}^{\lambda-1} \varepsilon \circ x^i y_1 F.$$

- We have for $[\tilde{\rho}_\lambda]_{11}$, $\lambda \leqslant 0$:

$$[\tilde{\rho}_\lambda]_{11} : EF_{\lambda+1}[y] \oplus A_{\lambda+1}[y]^{\oplus-\lambda} \to A_{\lambda+1}[y] \oplus FE_{\lambda+1}[y]$$

given by:

$$[\tilde{\rho}_\lambda]_{11} = \left(\left(\begin{smallmatrix} \varepsilon \\ \sigma \end{smallmatrix}\right), \sum_{i=0}^{-\lambda-1} \left(\begin{smallmatrix} y^i \\ Fh_{i-1}(x,y)\circ\eta \end{smallmatrix}\right) \right).$$

Proposition 5.9. *The morphism of $(A[y], A[y])$-bimodules $[\tilde{\rho}_\lambda]_{11}$ is an isomorphism for all λ.*

Proof. When $\lambda \geqslant 0$ and therefore $\lambda + 1 \geqslant 0$, the map:

$$\sigma \oplus \bigoplus_{i=0}^{\lambda} \varepsilon \circ x^i F : EF_{\lambda+1}[y] \xrightarrow{\sim} FE_{\lambda+1}[y] \oplus A_{\lambda+1}[y]^{\oplus\lambda+1}$$

is just $\rho_{\lambda+1} \otimes_k k[y]$. It is an isomorphism because $\rho_{\lambda+1}$ is an isomorphism.

Claim 5.10. When $\lambda \geqslant 0$, the map

$$\sigma \oplus \varepsilon \oplus \bigoplus_{i=0}^{\lambda-1} \varepsilon \circ x^i y_1 F : EF_{\lambda+1}[y] \to FE_{\lambda+1}[y] \oplus A_{\lambda+1}[y]^{\oplus\lambda+1}$$

is also an isomorphism.

Proof. Let $M_{-y} \in \mathrm{End}_{A_{\lambda+1}[y]}\left(A_{\lambda+1}[y]^{\oplus \lambda+1}\right)$ be the endomorphism with matrix coefficients $[M_{-y}] \in \mathrm{Mat}_{(\lambda+1)\times(\lambda+1)}\left(A_{\lambda+1}[y]^{\mathsf{op}}\right)$ given by 1 on the diagonal and $-y$ on the subdiagonal, and 0 elsewhere. This matrix is invertible, and M_{-y} is an isomorphism. Observe that:

$$\varepsilon \circ (-x^{i-1}yF) = -y \cdot \varepsilon \circ x^{i-1}F.$$

Using this we write the map in question as a composition of isomorphisms:

$$\sigma \oplus \varepsilon \oplus \bigoplus_{i=0}^{\lambda-1} \varepsilon \circ x^i y_1 F = \begin{pmatrix} 1 & 0 \\ 0 & M_{-y} \end{pmatrix} \circ \left(\sigma \oplus \bigoplus_{i=0}^{\lambda} \varepsilon \circ x^i F \right).$$

By reordering the first two summands in the codomain, we obtain the map $[\tilde{\rho}_\lambda]_{11}$. □

When $\lambda = 0$, the two formulas for $[\tilde{\rho}_\lambda]_{11}$ agree. Now assume $\lambda < 0$, so $\lambda + 1 \leqslant 0$ and the map:

$$\left(\sigma, \sum_{i=0}^{-(\lambda+1)-1} Fx^i \circ \eta \right) : EF_{\lambda+1} \oplus A_{\lambda+1}[y]^{\oplus -(\lambda+1)} \xrightarrow{\sim} FE_{\lambda+1}[y] \tag{5.6}$$

is $\rho_{\lambda+1} \otimes_k k[y]$, an isomorphism.

Claim 5.11. When $\lambda < 0$, the map:

$$\left(\sigma, \sum_{i=1}^{-\lambda-1} Fh_{i-1}(x,y) \circ \eta \right) : EF_{\lambda+1}[y] \oplus A_{\lambda+1}[y]^{\oplus -(\lambda+1)} \to FE_{\lambda+1}[y]$$

is also an isomorphism.

Proof. This time we define an isomorphism $M_h \in \mathrm{End}_{A_{\lambda+1}[y]}\left(A_{\lambda+1}[y]^{\oplus -(\lambda+1)}\right)$ with components $[M_h]_{ii} = 1$, $[M_h]_{ij} = y^{j-i}$ for $j > i$, and $[M_h]_{ij} = 0$ for $j < i$. This is an upper-triangular invertible matrix:

$$[M_h] = \begin{pmatrix} 1 & y & y^2 & \ldots & y^{-(\lambda+1)-1} \\ 0 & 1 & y & \ldots & y^{-(\lambda+1)-2} \\ 0 & 0 & 1 & \ldots & y^{-(\lambda+1)-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}.$$

Now observe that $Fx^i y^j \circ \eta = (Fx^i \circ \eta) \cdot y^j$. We use this and write:

$$\begin{aligned} \sum_{i=1}^{-\lambda-1} Fh_{i-1}(x,y) \circ \eta &= \sum_{i=0}^{-\lambda-2} \sum_{j+k=i} Fx^j \circ \eta \cdot y^k \\ &= \left(\sum_{i=0}^{-(\lambda+1)-1} Fx^i \circ \eta \right) \circ M_h, \end{aligned}$$

and it follows from this and the isomorphism above the claim that the map of the claim is an isomorphism. □

By writing out terms, we have:

$$\left(\begin{pmatrix} \varepsilon \\ \sigma \end{pmatrix}, \sum_{i=0}^{-\lambda-1} \begin{pmatrix} y^i \\ Fh_{i-1}(x,y)\circ\eta \end{pmatrix} \right) = \begin{pmatrix} \varepsilon & 1 & y & \ldots & y^{-\lambda-1} \\ \sigma & 0 & \eta & \ldots & Fh_{-\lambda-2}(x,y) \circ \eta \end{pmatrix}.$$

Interchanging the first two summands of the domain, we obtain the form:

$$\begin{pmatrix} 1 & \big(\varepsilon, y, y^2, \ldots, y^{-\lambda-1}\big) \\ 0 & \Big(\sigma, \sum_{i=1}^{-\lambda-1} Fh_{i-1}(x,y) \circ \eta\Big) \end{pmatrix},$$

which (by the claim) is manifestly an isomorphism. □

- We have for $[\tilde{\rho}_\lambda]_{21}$, $\lambda \geqslant 0$:

$$[\tilde{\rho}_\lambda]_{21} : F_{\lambda+1}[y] \oplus FEF_{\lambda+1}[y] \to F_{\lambda+1}[y] \oplus F_{\lambda+1}[y] \oplus F^2E_{\lambda+1}[y] \oplus F_{\lambda+1}[y]^{\oplus\lambda}$$

given by:

$$[\tilde{\rho}_\lambda]_{21} = \begin{pmatrix} 1 & 0 \\ 0 & F\varepsilon \\ 0 & F\sigma \\ \bigoplus_{i=0}^{\lambda-1} x^i & \bigoplus_{i=0}^{\lambda-1} F(\varepsilon \circ x^i y_1 F) \end{pmatrix}.$$

- We have for $[\tilde{\rho}_\lambda]_{21}$, $\lambda \leqslant 0$:

$$[\tilde{\rho}_\lambda]_{21} : F_{\lambda+1}[y] \oplus FEF_{\lambda+1}[y] \oplus F_{\lambda+1}[y]^{\oplus-\lambda} \to F_{\lambda+1}[y] \oplus F_{\lambda+1}[y] \oplus F^2E_{\lambda+1}[y]$$

given by:

$$[\tilde{\rho}_\lambda]_{21} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & F\varepsilon & \sum_{i=0}^{-\lambda-1} y^i \\ 0 & F\sigma & \sum_{i=0}^{-\lambda-1} F(Fh_{i-1}(x,y) \circ \eta) \end{pmatrix}.$$

Proposition 5.12. *The morphism of $(A[y], A[y])$-bimodules $[\tilde{\rho}_\lambda]_{21}$ is an isomorphism for all λ.*

Proof. When $\lambda \geqslant 0$, we have that

$$F\varepsilon \oplus F\sigma \oplus \bigoplus_{i=0}^{\lambda-1} F(\varepsilon \circ x^i y_1 F) : FEF_{\lambda+1}[y] \to F_{\lambda+1}[y] \oplus F^2E_{\lambda+1}[y] \oplus F_{\lambda+1}[y]^{\oplus\lambda}$$

is an isomorphism, using Claim 5.10 and the fact that (horizontal) composition of the identity functor on F with an isomorphism gives an isomorphism. Then $[\tilde{\rho}_\lambda]_{21}$ may be compressed to a lower-triangular 2×2 matrix with an isomorphism in position $(2,2)$, so it is an isomorphism.

When $\lambda = 0$, the two formulas for $[\tilde{\rho}_\lambda]_{21}$ agree. Assume now that $\lambda < 0$, so the map

$$\left(F\sigma, \sum_{i=1}^{-\lambda-1} F\big(Fh_{i-1}(x,y) \circ \eta\big)\right) : FEF_{\lambda+1}[y] \oplus F_{\lambda+1}[y]^{\oplus-(\lambda+1)} \to F^2E_{\lambda+1}[y]$$

is an isomorphism using Claim 5.11. Now expand the notation of the map $[\tilde{\rho}_\lambda]_{21}$ in the third column:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & F\varepsilon & 1 & \sum_{i=1}^{-\lambda-1} y^i \\ 0 & F\sigma & 0 & \sum_{i=1}^{-\lambda-1} F\big(Fh_{i-1}(x,y)\circ\eta\big) \end{pmatrix}.$$

After switching the second and third summands of the domain, we obtain an upper-triangular matrix with isomorphisms on the diagonal, so $[\tilde{\rho}_\lambda]_{21}$ is an isomorphism. □

- We have for $[\tilde{\rho}_\lambda]_{12}$, $\lambda \geqslant 0$:

$$[\tilde{\rho}_\lambda]_{12} : E_{\lambda-1}[y] \oplus EFE_{\lambda-1}[y] \to E_{\lambda-1}[y] \oplus E_{\lambda-1}[y] \oplus FE^2_{\lambda-1}[y] \oplus E_{\lambda-1}[y]^{\oplus\lambda}$$

given by:

$$[\tilde{\rho}_\lambda]_{12} = \begin{pmatrix} 0 & \varepsilon E \\ 1 & y_1 \circ \varepsilon E \\ 0 & \sigma E \\ \bigoplus_{i=0}^{\lambda-1} x^i & \bigoplus_{i=0}^{\lambda-1} (\varepsilon \circ x^i y_1 F)E \end{pmatrix}.$$

- We have for $[\tilde{\rho}_\lambda]_{12}$, $\lambda \leqslant 0$:

$$[\tilde{\rho}_\lambda]_{12} : E_{\lambda-1}[y] \oplus EFE_{\lambda-1}[y] \oplus E_{\lambda-1}[y]^{\oplus-\lambda} \to E_{\lambda-1}[y] \oplus E_{\lambda-1}[y] \oplus FE^2_{\lambda-1}[y]$$

given by:

$$[\tilde{\rho}_\lambda]_{12} = \begin{pmatrix} 0 & \varepsilon E & \sum_{i=0}^{-\lambda-1} y^i \\ 1 & y_1 \circ \varepsilon E & \sum_{i=0}^{-\lambda-1} y^i y_1 \\ 0 & \sigma E & \sum_{i=0}^{-\lambda-1} (Fh_{i-1}(x,y)\circ\eta)E \end{pmatrix}.$$

Proposition 5.13. *The morphism of $(A[y], A[y])$-bimodules $[\tilde{\rho}_\lambda]_{12}$ is an isomorphism for all λ.*

Proof. When $\lambda \geqslant 0$, we have that

$$\varepsilon E \oplus \sigma E \oplus \bigoplus_{i=0}^{\lambda-1} (\varepsilon \circ x^i y_1 F)E : EFE_{\lambda-1}[y] \to E_{\lambda-1}[y] \oplus FE^2_{\lambda-1}[y] \oplus E_{\lambda-1}[y]^{\oplus\lambda}$$

is an isomorphism, using Claim 5.10 with E applied on the right. Note that E applied on the right here is equivalent to ${}_{\lambda+1}E_{\lambda-1}$ applied on the right, and this raises the weight by 2, so we still invoke the isomorphism $\rho_{\lambda+1}$ for weight $\lambda + 1$.

We perform some row operations on the matrix of $[\tilde{\rho}_\lambda]_{12}$. Subtract y_1 times the first row from the second to eliminate the coefficient $y_1 \circ \varepsilon E$. Then exchange the first and second rows, then exchange the second and third rows, then collapse the second and third into the notation of the fourth. Obtain:

$$\begin{pmatrix} 1 & 0 \\ 0 \oplus 0 \oplus \bigoplus_{i=0}^{\lambda-1} x^i & \sigma E \oplus \varepsilon E \oplus \bigoplus_{i=0}^{\lambda-1} (\varepsilon \circ x^i y_1 F) E \end{pmatrix},$$

which is upper-triangular with isomorphisms on the diagonal, so the original matrix for $[\tilde{\rho}_\lambda]_{12}$ is an isomorphism.

When $\lambda = 0$, the two formulas for $[\tilde{\rho}_\lambda]_{12}$ agree. Assume now that $\lambda < 0$, so the map

$$\left(\sigma E, \sum_{i=1}^{-\lambda-1} \big(F h_{i-1}(x,y) \circ \eta\big) E\right) : EFE_{\lambda-1}[y] \oplus E_{\lambda-1}[y]^{\oplus -(\lambda+1)} \to FE_{\lambda-1}^2[y]$$

is an isomorphism using Claim 5.11. Now expand the notation of the map $[\tilde{\rho}_\lambda]_{12}$ in the third column:

$$[\tilde{\rho}_\lambda]_{12} = \begin{pmatrix} 0 & \varepsilon E & 1 & \sum_{i=1}^{-\lambda-1} y^i \\ 1 & y_1 \circ \varepsilon E & y_1 & \sum_{i=1}^{-\lambda-1} y^i y_1 \\ 0 & \sigma E & 0 & \sum_{i=1}^{-\lambda-1} \big(F h_{i-1}(x,y) \circ \eta\big) E \end{pmatrix}.$$

Exchange the first and second rows, then the second and third columns, then collapse the third and fourth columns into the notation of the third, and obtain:

$$\begin{pmatrix} 1 & y_1 & \left(y_1 \circ \varepsilon E, \sum_{i=1}^{-\lambda-1} y^i y_1\right) \\ 0 & 1 & \left(\varepsilon E, \sum_{i=1}^{-\lambda-1} y^i\right) \\ 0 & 0 & \left(\sigma E, \sum_{i=1}^{-\lambda-1} \big(F h_{i-1}(x,y) \circ \eta\big) E\right) \end{pmatrix}.$$

Since this is upper-triangular with isomorphisms on the diagonal, the original matrix $[\tilde{\rho}_\lambda]_{12}$ is an isomorphism. □

- We have for $[\tilde{\rho}_\lambda]_{22}$, $\lambda \geqslant 0$:

$$(5.7) \quad [\tilde{\rho}_\lambda]_{22} : A_{\lambda-1}[y] \oplus FE_{\lambda-1}[y]^{\oplus 2} \oplus FEFE_{\lambda-1}[y] \oplus EF_{\lambda-1}[y] \\ \to FE_{\lambda-1}[y]^{\oplus 4} \oplus F^2E_{\lambda-1}^2[y] \oplus A_{\lambda-1}[y]^{\oplus \lambda} \oplus FE_{\lambda-1}[y]^{\oplus \lambda}$$

given by: $[\tilde{\rho}_\lambda]_{22} =$

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & F\varepsilon E & 0 \\ \eta & y_1 & 0 & 0 & \sigma \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & F\sigma E & 0 \\ \bigoplus_{i=0}^{\lambda-1} y^i & 0 & 0 & 0 & \bigoplus_{i=0}^{\lambda-1} -\varepsilon \circ h_{i-1}(x,y)F \\ \bigoplus_{i=0}^{\lambda-1} h_{i-1}(x,y)\circ\eta & \bigoplus_{i=0}^{\lambda-1} x^iE & \bigoplus_{i=0}^{\lambda-1} Fx^i & \bigoplus_{i=0}^{\lambda-1} F(\varepsilon\circ x^i y_1 F)E & \Theta \end{pmatrix},$$

where

$$\Theta = \bigoplus_{i=0}^{\lambda-1} -FE\varepsilon \circ F\big(\tau \circ h_{i-1}(x_1,x_2) - h_{i-2}(x_1,x_2,y)\big)F \circ \eta EF.$$

- We have for $[\tilde{\rho}_\lambda]_{22}$, $\lambda \leqslant 0$:

$$(5.8)\quad \begin{aligned} [\tilde{\rho}_\lambda]_{22} : A_{\lambda-1}[y] \oplus FE_{\lambda-1}[y]^{\oplus 2} \oplus FEFE_{\lambda-1}[y] \oplus EF_{\lambda-1}[y] \\ \oplus A_{\lambda-1}[y]^{\oplus-\lambda} \oplus FE_{\lambda-1}[y]^{\oplus-\lambda} \to FE_{\lambda-1}[y]^{\oplus 4} \oplus F^2E^2_{\lambda-1}[y] \end{aligned}$$

given by: $[\tilde{\rho}_\lambda]_{22} =$

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & \sum_{i=0}^{-\lambda-1} Fy^i \circ \eta & \sum_{i=0}^{-\lambda-1} y^i y_1 \\ 0 & 0 & 0 & F\varepsilon E & 0 & \sum_{i=0}^{-\lambda-1} -Fh_{i-1}(x,y)\circ\eta & \sum_{i=0}^{-\lambda-1} y^i \\ \eta & y_1 & 0 & 0 & \sigma & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \sum_{i=0}^{-\lambda-1} Fx^i \circ \eta & 0 \\ 0 & 0 & 0 & F\sigma E & 0 & \Theta' & \sum_{i=0}^{-\lambda-1} F^2\big(h_{i-1}(x_2,y)\big)\circ F\eta E \end{pmatrix},$$

where

$$\Theta' = \sum_{i=0}^{-\lambda-1} F^2\big(h_{i-1}(x_1,x_2)\circ\tau - h_{i-2}(x_1,x_2,y)\big)\circ\eta^2.$$

Proposition 5.14. *The morphism of $(A[y], A[y])$-bimodules $[\tilde{\rho}_\lambda]_{22}$ is an isomorphism for all λ.*

Proof. When $\lambda > 0$ and therefore $\lambda - 1 \geqslant 0$, the map

$$\sigma \oplus \bigoplus_{i=0}^{\lambda-2} -\varepsilon \circ x^i F : EF_{\lambda-1}[y] \to FE_{\lambda-1}[y] \oplus A_{\lambda-1}[y]^{\oplus\lambda-1}$$

is an isomorphism. (The minus sign does not interfere.)

Claim 5.15. When $\lambda > 0$, the map

$$\sigma \oplus \bigoplus_{i=1}^{\lambda-1} -\varepsilon \circ h_{i-1}(x,y)F : EF_{\lambda-1}[y] \to FE_{\lambda-1}[y] \oplus A_{\lambda-1}[y]^{\oplus\lambda-1}$$

is an isomorphism.

Proof. Define an isomorphism $M'_h \in \mathrm{End}_{A_{\lambda-1}[y]}\left(A_{\lambda-1}[y]^{\oplus\lambda-1}\right)$ with components $[M'_h]_{ii} = 1$, $[M'_h]_{ij} = y^{i-j}$ for $i > j$, and $[M'_h]_{ij} = 0$ for $i < j$. This is a lower-triangular invertible matrix:

$$[M'_h] = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ y & 1 & 0 & \dots & 0 \\ y^2 & y & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ y^{\lambda-2} & y^{\lambda-3} & y^{\lambda-4} & \dots & 1 \end{pmatrix}.$$

Now observe that $\varepsilon \circ x^i y^j F = y^j \cdot \varepsilon \circ x^i F$. Using this, we can write:

$$\begin{aligned} \bigoplus_{i=1}^{\lambda-1} -\varepsilon \circ h_{i-1}(x,y)F &= \bigoplus_{i=0}^{\lambda-2} \sum_{j+k=i} y^k \cdot (-\varepsilon \circ x^j F) \\ &= M'_h \circ \left(\bigoplus_{i=0}^{\lambda-2} -\varepsilon \circ x^j F \right), \end{aligned}$$

and it follows from this and the isomorphism above the claim that the map of the claim is an isomorphism. □

Now assume $\lambda > 0$ and reorder the summands of the domain and codomain to permute the rows and columns of the matrix of $[\tilde{\rho}_\lambda]_{22}$. Let the domain be given in the order:

$$FE_{\lambda-1}[y]^{\oplus 2} \oplus A_{\lambda-1}[y] \oplus EF_{\lambda-1}[y] \oplus FEFE_{\lambda-1}[y],$$

where the first two identical summands appear in the same order as before. Let the codomain be given in the order:

$$\begin{aligned} FE_{\lambda-1}[y]^{\oplus 2} \oplus A_{\lambda-1}[y] \oplus FE_{\lambda-1}[y] \oplus A_{\lambda-1}[y]^{\oplus\lambda-1} \\ \oplus F^2E^2_{\lambda-1}[y] \oplus FE_{\lambda-1}[y] \oplus FE_{\lambda-1}[y]^{\oplus\lambda-1}, \end{aligned}$$

where the new summand number (numbered left to right) and corresponding old summand number are given precisely in the following chart:

new:	1	2	3	4	5	6	7	...	$\lambda+3$	$\lambda+4$	$\lambda+5$	$\lambda+6$	...	$2\lambda+5$
old:	4	1	6	3	7	8	9	...	$\lambda+5$	2	5	$\lambda+6$	...	$2\lambda+5$.

Writing the matrix of $[\tilde{\rho}_\lambda]_{22}$ for $\lambda > 0$, with columns and rows changed by the above permutations, we obtain:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ y_1 & 0 & \eta & \sigma & 0 \\ 0 & 0 & \bigoplus_{i=1}^{\lambda-1} y^i & \bigoplus_{i=1}^{\lambda-1} -\varepsilon \circ h_{i-1}(x,y)F & 0 \\ 0 & 0 & 0 & 0 & F\varepsilon E \\ 0 & 0 & 0 & 0 & F\sigma E \\ \bigoplus_{i=0}^{\lambda-1} x^i E & \bigoplus_{i=0}^{\lambda-1} Fx^i & \bigoplus_{i=0}^{\lambda-1} h_{i-1}(x,y) \circ \eta & \Theta & \bigoplus_{i=0}^{\lambda-1} F(\varepsilon \circ x^i y_1 F)E \end{pmatrix}.$$

After compressing the notation of rows 4-5 and 6-8 of this matrix, we obtain a lower-triangular matrix. The last two diagonal entries are:

$$\begin{pmatrix} \sigma \\ \bigoplus_{i=1}^{\lambda-1} -\varepsilon \circ h_{i-1}(x,y)F \end{pmatrix},$$

which is an isomorphism by the claim, and:

$$\begin{pmatrix} F\varepsilon E \\ F\sigma E \\ \bigoplus_{i=0}^{\lambda-1} F(\varepsilon \circ x^i y_1 F)E \end{pmatrix} : FEFE_{\lambda-1}[y] \to FE_{\lambda-1}[y] \oplus F^2E^2_{\lambda-1}[y] \oplus FE_{\lambda-1}[y]^{\oplus\lambda},$$

which is an isomorphism for $\lambda > 0$, and therefore for $\lambda+1 \geqslant 0$, using Claim 5.10 with F applied on the left and E on the right.

When $\lambda = 0$ the matrix of $[\tilde{\rho}_\lambda]_{22}$ is given by removing rows 3, 5–$(\lambda+3)$, and $(\lambda+6)$–$(2\lambda+5)$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ y_1 & 0 & \eta & \sigma & 0 \\ 0 & 0 & 0 & 0 & F\varepsilon E \\ 0 & 0 & 0 & 0 & F\sigma E \end{pmatrix}.$$

When $\lambda = 0$ we also have isomorphisms:

$$(\eta, \sigma) : EF_{\lambda-1}[y] \oplus A[y]_{\lambda-1} \xrightarrow{\sim} FE_{\lambda-1}[y]$$

and

$$\left(\begin{smallmatrix} F\varepsilon E \\ F\sigma E \end{smallmatrix} \right) : F\big(FE_{\lambda+1}\big)E[y] \xrightarrow{\sim} F_{\lambda+1}E[y] \oplus F\big(EF_{\lambda+1}\big)E[y],$$

so we see that again the matrix can be written as a lower-triangular matrix with invertible diagonal entries.

Finally, assume $\lambda < 0$. We have an isomorphism:

$$\left(\sigma, \sum_{i=0}^{-\lambda} Fx^i \circ \eta\right) : EF_{\lambda-1}[y] \oplus A_{\lambda-1}[y]^{\oplus-(\lambda-1)} \xrightarrow{\sim} FE_{\lambda-1}[y],$$

which is the isomorphism $\rho_{\lambda-1} \otimes_k k[y]$. There is a final claim to check:

Claim 5.16. When $\lambda < 0$, the map

$$\left(\sigma, \eta, \sum_{i=0}^{-\lambda-1} -Fx^i y_1 \circ \eta\right) : EF_{\lambda-1}[y] \oplus A_{\lambda-1}[y]^{\oplus-(\lambda-1)} \to FE_{\lambda-1}[y]$$

is an isomorphism.

Proof. Define an isomorphism $M'_{-y} \in \mathrm{End}_{A_{\lambda-1}[y]}\left(A_{\lambda-1}[y]^{\oplus-(\lambda-1)}\right)$ with components $[M_h]_{ij}$ given by 1 along the diagonal and $-y$ along the subdiagonal. This is a lower-triangular invertible matrix. We write the map in question as a composition of isomorphisms:

$$\left(\sigma, \eta, \sum_{i=0}^{-\lambda-1} -Fx^i y_1 \circ \eta\right) = \left(\sigma, \eta, \sum_{i=1}^{-\lambda} Fx^i \circ \eta\right)$$
$$\circ \begin{pmatrix} \mathrm{Id}_{EF_{\lambda-1}[y]} & 0 & 0 \\ 0 & \mathrm{Id}_{A_{\lambda-1}[y]} & 0 \\ 0 & 0 & -\mathrm{Id}_{A_{\lambda-1}[y]^{\oplus -\lambda}} \end{pmatrix} \circ \begin{pmatrix} \mathrm{Id}_{EF_{\lambda-1}[y]} & 0 \\ 0 & M'_{-y} \end{pmatrix}.$$

□

Now let W be the endomorphism of the codomain of $[\tilde{\rho}_\lambda]_{22}$ given by the invertible matrix:

$$[W] = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -y_1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We show that $[W] \cdot [\tilde{\rho}_\lambda]_{22}$ is equivalent to a lower-triangular matrix after giving a suitable permutation of the domain and codomain summands. Let the domain be given in the order:

$$EF_{\lambda-1}[y] \oplus A_{\lambda-1}[y]^{-(\lambda-1)} \oplus FE_{\lambda-1}[y] \oplus FEFE_{\lambda-1}[y] \oplus FE_{\lambda-1}[y]^{-(\lambda+1)} \oplus FE_{\lambda-1}[y]^{\oplus 2},$$

where the change of summand numbers is given by the following chart:

new:	1	2	3	4	...	$-\lambda+2$	$-\lambda+3$	$-\lambda+4$
old:	5	1	6	7	...	$-\lambda+5$	2	4

new:	$-\lambda+5$	$-\lambda+6$	...	$-2\lambda+4$	$-2\lambda+5$	$-2\lambda+6$
old:	$-\lambda+7$	$-\lambda+8$	...	$-2\lambda+5$	$-\lambda+6$	3.

Let the codomain be given in the order:

$$FE_{\lambda-1}[y]^{\oplus 4} \oplus F^2E^2_{\lambda-1}[y],$$

where the change of summand numbers is given by the following chart:

new:	1	2	3	4	5
old:	3	4	5	2	1.

The matrix of $[W] \cdot [\tilde{\rho}_\lambda]_{22}$ for $\lambda < 0$ agrees with that for $[\tilde{\rho}_\lambda]_{22}$ except in the third row, where it is:

$$\left(\eta \quad 0 \quad 0 \quad 0 \quad \sigma \quad \sum_{i=0}^{-\lambda-1} -Fx^i y_1 \circ \eta \quad 0 \quad 0\right).$$

Writing now the matrix of $[W] \cdot [\tilde{\rho}_\lambda]_{22}$ with columns and rows changed by the above permutations, and compressing the notation for some columns, we obtain:

$$\begin{pmatrix} \left(\sigma,\eta,\sum_{i=0}^{-\lambda-1}-Fx^iy_1\circ\eta\right) & 0 & (0,0) & 0 & 0 \\ \left(0,0,\sum_{i=0}^{-\lambda-1}Fx^i\circ\eta\right) & 1 & (0,0) & 0 & 0 \\ (0,0,\Theta') & 0 & \left(F\sigma E,\sum_{i=1}^{-\lambda-1}F^2h_{i-1}(x_2,y)\circ F\eta E\right) & 0 & 0 \\ \left(0,0,\sum_{i=0}^{-\lambda-1}-Fh_{i-1}(x,y)\circ\eta\right) & 0 & \left(F\varepsilon E,\sum_{i=1}^{-\lambda-1}y^i\right) & 1 & 0 \\ \left(0,0,\sum_{i=0}^{-\lambda-1}Fy^i\circ\eta\right) & 0 & \left(0,\sum_{i=1}^{-\lambda-1}y^iy_1\right) & y_1 & 1 \end{pmatrix}.$$

The upper left map is an isomorphism by the Claim proved above. The middle diagonal map is an isomorphism because it is the isomorphism of Claim 5.11 with F applied on the left and E on the right. So the matrix is lower-triangular with isomorphisms along the diagonal. □

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