A TENSOR 2-PRODUCT OF 2-REPRESENTATIONS OF $\mathfrak{sl}_2^+$

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Abstract. We construct an explicit abelian model for the operation of tensor 2-product of 2-representations of $\mathfrak{sl}_2^+$, specifically the product of a simple 2-representation $\mathcal{L}(1)$ with a given abelian 2-representation $\mathcal{V}$ taken from the 2-category of algebras. We study the case $\mathcal{V} = \mathcal{L}(1)$ in detail, and we show that the 2-product in this case recovers the expected structure. Our construction partially verifies a conjecture of Rouquier from 2008.

Contents

1. Introduction 2
1.1. Background and motivation 2
1.2. Result 3
1.3. Technique 4
1.4. Explanation 5
1.5. Outline summary 6
1.6. Acknowledgments 6
2. Background structures 6
2.1. Nil affine Hecke algebras 6
2.2. $\mathcal{U}^+(\mathfrak{sl}_2)$ and its 2-representations 6
2.3. Further conventions 8
2.4. Generalized matrix algebras and tensor product 10
3. Product category 11
3.1. Naive product category 11
3.2. Bimodules G_n 16
3.3. Product category C-mod 23
3.4. New bimodule $\tilde{E}$ 26
4. Hecke action 28
4.1. Definition of the action 28
4.2. Hecke relations 36
4.3. Definition of $\mathcal{L}(1) \bigotimes \mathcal{V}$ 43
5. Comparison: $\mathcal{V} = \mathcal{L}(1)$ 44
5.1. A categorification of $L(1) \otimes L(1)$ 45
5.2. $\mathcal{L}(1) \bigotimes \mathcal{L}(1)$ 46
5.3. Comparison 47
References 49

1. Introduction

1.1. **Background and motivation.** The operation of tensor product is ubiquitous in representation theory and its applications. It is a primary means of generating new representations from old ones. In classical Lie theory this operation arises from the Hopf structure of the enveloping algebra.

In [CF94], Crane and Frenkel outlined a program to build topological invariants using a higher representation theory. The program was conceived as a way to formulate invariants algebraically in $4d$ that upgrade known invariants in $3d$ such as the TQFT of Witten-Reshetikhin-Turaev [Wit89, RT91]. The idea was to implement categorical versions of classical algebraic structures. Crane and Frenkel proposed a concept of 'Hopf category' to upgrade the Hopf structure of quantum groups that was central to the WRT invariant.

A fully developed Hopf categorical representation theory will have good definitions of categorical algebra, categorical representation, and categorical Hopf structure. The notion of 2-representation was provided with a good definition for $\mathfrak{sl}_2$ in work of Chuang-Rouquier [CR08], and the graded case descending to quantized structures in work of Lauda [Lau10]. The definitions were generalized to all Kac-Moody algebras in [Rou08a] and [KL09, KL11].

A tensor 2-product operation would give the higher analog of an aspect of Hopf structure, or at least of the expression of Hopf structure on the category of representations of the algebra. A 2-product is defined in an $\mathcal{A}_\infty$ setting by Rouquier [Rou], but no explicit formulas are known for the product action in that setting, and the setting itself brings significant technical complications. Rouquier has conjectured [Rou08b] that a subcategory affording an abelian 2-representation should exist. The main construction of this paper partially verifies his conjecture by identifying an abelian 2-product when one factor is $\mathcal{L}(1)$ and the other factor $\mathcal{V}$ is taken from the 2-category of algebras. In addition, our construction takes a step toward defining a practicable 2-product by providing explicit formulas for the 2-representation component structures.

In early work of Bernstein-Frenkel-Khovanov [BFK99], the authors consider a category whose Grothendieck group is the tensor product of fundamental representations. Their methods were extended by Stroppel [Str05] and others (cf. [FKS07, MS09, SS15, Sus07]) to find a category with Grothendieck group isomorphic to any given tensor product of finite dimensional simples in type A. Graphical methods were developed by Webster [Web17, Web16] to produce categories for tensor products of simples for general Kac-Moody algebras. We expect these categories to be equivalent to tensor 2-products of simple 2-representations.

The Crane-Frenkel program for building TQFTs gives perhaps the most compelling motivation to find a categorical product. In the case of $\mathfrak{sl}_2$, a 2-product will play a central role in a prospective $4d$ TQFT that extends Khovanov homology. Glimmers of this $4d$ theory have been seen by physicists [GPV17], and some aspects are defined rigorously in some cases [GM21]. Along these lines, recent work of Manion-Rouquier [MR20] on the case of the super Lie algebra $\mathfrak{gl}(1|1)^+$ shows that a 2-product can be used to describe Bordered Heegaard-Floer theory for surfaces [LOT18].

1.2. **Result.** Let $\mathcal{U}^+$ denote the monoidal category associated to the positive half of the enveloping algebra of $\mathfrak{sl}_2$. Let us be given a field k and the data of a k-algebra A and a triple (E, x, τ) as follows. Let E be an (A, A)-bimodule, let $x \in \mathrm{End}(E)$ and $\tau \in \mathrm{End}(E^2)$ be bimodule endomorphisms, and suppose that x and τ generate an action of the nil affine Hecke algebra, that is, that they satisfy the following relations:

$$\tau^2 = 0,$$
$$\tau E \circ E\tau \circ \tau E = E\tau \circ \tau E \circ E\tau,$$
$$\tau \circ Ex = xE \circ \tau + 1, \; Ex \circ \tau = \tau \circ xE + 1.$$

(Here we write xE for the endomorphism $x \otimes \mathrm{Id}_E$ in $\mathrm{End}(E^2)$, and similarly for the others.) This data determines a 2-representation $\mathcal{V}$ of $\mathcal{U}^+$.

We can give such data for a simple 2-representation $\mathcal{L}(1)$ of $\mathcal{U}^+$ that categorifies the fundamental representation $L(1)$ of $\mathfrak{sl}_2$. The k-algebra is $k[y]_{+1} \times k[y]_{-1}$ (decomposed into weight algebras), and the triple is $(k[y], y, 0)$. Here $y \in k[y]_{-1}$ acts on $k[y]$ on the right by multiplication, and $y \in k[y]_{+1}$ acts by zero. These roles are reversed for the left action. The endomorphism x acts by multiplication by y.

Let $P_n = k[x_1, \ldots, x_n]$ be the polynomial algebra. Then P_n acts on E^n with $x_i \in P_n$ acting by the endomorphism $E^{n-i}xE^{i-1}$.

This paper is organized around a proof of the following theorem.

Theorem (Main result)**.** *Suppose x and τ satisfy the nil affine Hecke relations, so (E, x, τ) gives a 2-representation of $\mathcal{U}^+$ for the algebra A, denoted $\mathcal{V}$, and suppose the bimodule E has the following additional properties:*

- *${}_AE$ is finitely generated and projective,*
- *E^n is free as a P_n-module.*

Then we define explicitly:

- *a k-algebra C (Def. 3.32),*
- *a bimodule $\tilde{E}$ (Def. 3.38),*
- *endomorphisms $\tilde{x}$ and $\tilde{\tau}$ (Def. 4.4),*

such that $\tilde{x}$ and $\tilde{\tau}$ satisfy the nil affine Hecke relations, so $(\tilde{E}, \tilde{x}, \tilde{\tau})$ gives the data of a 2-representation of $\mathcal{U}^+$ for C that we denote $\mathcal{L}(1) \bigotimes \mathcal{V}$.

We have two reasons to interpret the new 2-representation as an abelian model for the 2-product $\mathcal{L}(1) \bigotimes \mathcal{V}$: it is derived from an approach to categorifying the Hopf coproduct formula, and in a class of cases it recovers the expected result. In this document we study the case $\mathcal{L}(1) \bigotimes \mathcal{L}(1)$ in detail. In forthcoming work with Laurent Vera we show that $\mathcal{L}(1) \bigotimes \mathcal{L}(n)$ recovers the expected structure for every $n \in \mathbb{Z}^{>0}$.

In another paper [McM23] we consider the extension of the construction given in this paper to actions of the full 2-category $\mathcal{U}$ associated to the enveloping algebra of $\mathfrak{sl}_2$, and not only its positive half. When the functor $E \otimes_A -$ has a right adjoint given by tensor product with a bimodule F, and the pair of them satisfies some additional relations that categorify the commutator identities, the action is said to give a 2-representation of $\mathcal{U}$. We show that if the original bimodule E has such an F giving an action of the full $\mathcal{U}$ on $\mathcal{V}$, then there is also a bimodule $\tilde{F}$, given as the right-dual of $\tilde{E}$, which together with $\tilde{E}$ provides an action of the full $\mathcal{U}$ on $\mathcal{L}(1) \bigotimes \mathcal{V}$.

In a third paper (forthcoming) we consider several questions about the 2-product construction that are motivated by the search for a $4d$ TQFT. For example, one would like to iterate the construction:

$$\mathcal{L}(1)^{\bigotimes n} = \mathcal{L}(1) \bigotimes \Big(\mathcal{L}(1) \bigotimes \big(\mathcal{L}(1) \bigotimes \ldots\big)\Big).$$

To define this product, we need to establish that our $\tilde{E}^n$ construction is free as a $k[\tilde{x}_1, \ldots, \tilde{x}_n]$-module. We also want a product in the reverse order, $\mathcal{V} \bigotimes \mathcal{L}(1)$, to determine an iterated product with arbitrary parenthesization. Questions about associativity make sense at that point. We would like to establish functoriality in the argument $\mathcal{V}$. A further step would be to produce a braid group action on iterates $\mathcal{L}(1)^{\bigotimes n}$, as well as 'cup and cap' morphisms.

1.3. **Technique.** Let us be given $\mathcal{V}$ as described above. Write E_y for the $(A[y], A[y])$-bimodule $E[y]\big/(x-y)E[y]$. We begin with a 'naive' algebra B formed from the underlying data of $\mathcal{L}(1)$ and $\mathcal{V}$:

$$B = \begin{pmatrix} A[y] & E_y \\ 0 & A[y] \end{pmatrix}.$$

There is a natural candidate E' for the diagonal action of $\mathcal{U}^+$, but it is a complex of (B, B)-bimodules, not a bimodule. It is given as a complex in degrees 0 and 1 by

$$E' = \begin{pmatrix} E[y] & E[y]E_y \\ 0 & E[y] \end{pmatrix} \xrightarrow{d} \begin{pmatrix} E_y & E_yE_y \\ A[y] & E_y \end{pmatrix}.$$

(The differential and action data are described in Definition 3.2.) There is also a natural candidate for $x \in \operatorname{End}(E')$ arising from the data of $\mathcal{L}(1)$ and $\mathcal{V}$, but that x is not equivariant over the action of generators in E_y in B.

Let $e_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in B$. Our technique in this paper is to define a new algebra

$$C = \operatorname{End}_{K^b(B)}(Be_1 \oplus E'e_1)$$

that is derived-equivalent to B. The bimodule complex E' may be transported through the equivalence, and the result is quasi-isomorphic to a complex $\tilde{E}$ of (C, C)-bimodules that is concentrated in degree 0 and projective on the left. We consider $\tilde{E}$ to be a (C, C)-bimodule, and we construct explicit bimodule endomorphisms $\tilde{x} \in \operatorname{End}(\tilde{E})$ (compatible with x) and $\tilde{\tau} \in \operatorname{End}(\tilde{E}^2)$ that satisfy the nil affine Hecke relations. The data $(C, \tilde{E}, \tilde{x}, \tilde{\tau})$ determines a 2-representation that we call $\mathcal{L}(1) \bigotimes \mathcal{V}$.

In order to define $\tilde{x}$ and $\tilde{\tau}$ and verify the relations, we study the tensor powers $\tilde{E}^n$. These powers can be parametrized by explicit models containing $\operatorname{Hom}_{K^b(B)}(E'e_1, E'^n e_1)$. We give presentations of these modules by generators and relations for $n = 1, 2, 3, 4$.

1.4. **Explanation.** Suppose ${}_HM$ and ${}_HN$ are two representations of a Hopf k-algebra H with coproduct $\Delta : H \to H^2$ and antipode $S : H \to H$. There is a large outer product $M \otimes_k N$ with two commuting actions of H on the two factors, and a third, diagonal, action given by first applying Δ. There is a smaller product $M \otimes_H N$ using S to view M as a right H-module. The smaller product is related to the larger one as follows: $M \otimes_H N$ is the largest quotient of $M \otimes_k N$ on which $\Delta(H)$ acts by 0. This can be seen using the formulas $\Delta(h) = h \otimes 1 + 1 \otimes h$ and $S(h) = -h$ for enveloping algebras of Lie algebras, with which the condition $\Delta(h).(m \otimes n) = 0$ may be written $m.h \otimes n = m \otimes h.n$.

Now let $\mathcal{V}_i$ be an abelian category of A_i-modules for $i = 1, 2$, where $\mathcal{V}_i$ is a 2-representation of $\mathcal{U}^+$ with data (E_i, x^i, τ^i). We can easily define a large outer product category $\mathcal{V}_1 \boxtimes_k \mathcal{V}_2$ that has two commuting actions of $\mathcal{U}^+$. We seek a kind of diagonal action of $\mathcal{U}^+$ on $\mathcal{V}_1 \boxtimes_k \mathcal{V}_2$. One can also describe a smaller product without diagonal $\mathcal{U}^+$-symmetry. Objects should be generated by pairs of modules $M \in \mathcal{V}_1$, $N \in \mathcal{V}_2$ together with functorial isomorphisms $E_1(M) \otimes_k N \xrightarrow{\sim} M \otimes_k E_2(N)$ that are equivariant over the actions of x^i on E_i and τ^i on E_i^2. These isomorphisms categorify the conditions $\Delta(e).(m \otimes n) = 0$.

At this point we make three conceptual moves. First, we expand the larger product category by including with each pair $M \in \mathcal{V}_1$, $N \in \mathcal{V}_2$ a morphism $\alpha_M^N : E_1(M) \otimes_k N \to M \otimes_k E_2(N)$, functorial in M and N, that is x- and τ-equivariant. So we define objects of $\mathcal{V}_1 \circledcirc\!\!\!\!\otimes \mathcal{V}_2$ to be triples of the form $(M, N; \alpha_M^N)$. Second, we consider morphisms α_M^N as two-term chain complexes, in particular mapping cones, and move to a derived context. Third, for the new diagonal action of E on $(M, N; \alpha_M^N)$ we take the cone complex $C = Cone(\alpha_M^N)$ itself. In the derived category, this complex is zero precisely when α_M^N is an isomorphism, which is the correspondence we sought.

To complete the idea, it is necessary to supply natural x- and τ-equivariant morphisms $\alpha_{(E_1 \otimes \mathrm{Id})C}^{(\mathrm{Id} \otimes E_2)C}$ in order to make C an object in $\mathcal{V}_1 \circledcirc\!\!\!\!\otimes \mathcal{V}_2$, and to supply endomorphisms x and τ of $Cone(\alpha_M^N)$ and $Cone(Cone(\alpha_M^N))$ satisfying Hecke-type relations in order to make a 2-representation of $\mathcal{U}$ using $Cone(\alpha_M^N)$ for the image of E. Here one encounters further technical difficulties. In [Rou], Rouquier is expected to give a general definition of tensor 2-product by working in an $\mathcal{A}_\infty$ setting that encodes the technical difficulties as higher homotopies. For example, the failure of equivariance of the natural $x \in \mathrm{End}(E')$ mentioned in §1.3 can be expressed as a homotopy.

In our setting for $\mathcal{L}(1) \circledcirc\!\!\!\!\otimes \mathcal{V}$, we have $\mathcal{L}(1)$ given by the data $(A^\circ, k[y], y, 0)$ with $A^\circ = k[y]_{+1} \times k[y]_{-1}$, and $\mathcal{V}$ given by the data (A, E, x, τ). One can define a tensor algebra B':

$$B' = T_{A^\circ \otimes_k A}\big({}^\vee k[y] \otimes_k E\big).$$

There is a canonical isomorphism ${}^\vee k[y] \otimes_k E \xrightarrow{\sim} E[y]$, and another $A^\circ \otimes_k A \xrightarrow{\sim} A[y] \times A[y]$. The data of a B'-module is equivalent to the data of a triple (M, N, α_M^N) where $M, N \in A[y]$-mod and $\alpha_M^N : E[y] \otimes_{A[y]} M \to N$. Since $\tau^1 = 0$ in this case, α is automatically τ-equivariant. We can enforce x-equivariance of α by taking a quotient by $I = \mathrm{Im}(x - y)$, where $x - y$ is understood in $\mathrm{End}_{A[y]}(E[y])$. Then the algebra B'/I is isomorphic to the algebra B in §1.3.

1.5. **Outline summary.** The paper is organized as follows:

- In §2 we describe some conventions and background theory. We are working in the setting of monoidal categories of the form $\mathsf{Bim}_k(A)$ for a k-algebra A: objects are (A, A)-bimodules, morphisms are bimodule maps. The data of a 2-representation of $\mathcal{U}^+$ consists of an algebra A, a bimodule ${}_AE_A$, and endomorphisms $x \in \mathrm{End}(E)$ and $\tau \in \mathrm{End}(E^2)$ satisfying nil affine Hecke relations.
- In §3 we begin with a naive product algebra B and complex of bimodules ${}_BE'_B$. We construct a derived-equivalent algebra C. We define a (C, C)-bimodule $\tilde{E}$ and study a new class of bimodules we call G_n that arise inside the tensor powers of $\tilde{E}$. This study has a technical and computational flavor.
- In §4 we construct the new nil affine Hecke action, with generators $\tilde{x}$ and $\tilde{\tau}$, on powers of the new bimodule $\tilde{E}$. More computations are required to establish the properties we need. They rely on results about G_n proved in §3.
- In §5 we give explicit details for the most basic example of our construction: $\mathcal{L}(1) \circledR \mathcal{L}(1)$. This product agrees with a well-known categorification of $L(1) \otimes L(1)$, where $L(1)$ is the fundamental representation of $\mathfrak{sl}_2$.

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2. Background structures

Let k be a field.

2.1. **Nil affine Hecke algebras.** The nil affine Hecke algebra ${}^0H_n$ is the k-algebra with generators $x_1, \ldots, x_n, \tau_1, \ldots, \tau_{n-1}$ and relations:

$$x_i x_j = x_j x_i, \tau_i^2 = 0,$$
$$\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1},$$
$$\tau_i \tau_j = \tau_j \tau_i \text{ if } |i - j| > 1,$$
$$\tau_i x_j = x_j \tau_i \text{ if } j - i \notin \{0, 1\},$$
$$\tau_i x_i = x_{i+1} \tau_i + 1, x_i \tau_i = \tau_i x_{i+1} + 1.$$

Define $s_i = \tau_i(x_i - x_{i+1}) - 1$. Observe that $s_i^2 = 1$ and $s_i \circ \tau_i = \tau_i$.

2.2. $\mathcal{U}^+(\mathfrak{sl}_2)$ **and its 2-representations.**

2.2.1. *Monoidal category $\mathcal{U}^+$.*

Definition 2.1. Let $\mathcal{U}^+(\mathfrak{sl}_2)$ (hereafter '$\mathcal{U}^+$') be the strict monoidal k-linear category generated by an object E and maps $x : E \to E$ and $\tau : E^2 \to E^2$ subject to the relations:

$$\tau^2 = 0, \tag{2.1}$$

$$\tau E \circ E\tau \circ \tau E = E\tau \circ \tau E \circ E\tau, \tag{2.2}$$

$$\tau \circ Ex = xE \circ \tau + 1, \; Ex \circ \tau = \tau \circ xE + 1. \tag{2.3}$$

We write $s = \tau \circ (Ex - xE) - 1$. Observe that $s^2 = 1$ and $s \circ \tau = \tau$.

One easily checks that non-trivial Hom spaces of $\mathcal{U}^+$ are Hecke algebras:

Proposition 2.2. *The objects of $\mathcal{U}^+$ are the E^n for $n \in \mathbb{Z}^{\geqslant 0}$, and*

$$\mathrm{Hom}(E^n, E^m) \cong \begin{cases} {}^0H_n & n = m \\ 0 & n \neq m \end{cases}$$

with the isomorphism from ${}^0H_n$ given by $x_i \mapsto E^{n-i}xE^{i-1}$, $\tau_i \mapsto E^{n-i-1}\tau E^{i-1}$. Using the obvious morphism ${}^0H_n \otimes {}^0H_m \to {}^0H_{n+m}$, the diagram commutes:

$$\begin{array}{ccc} {}^0H_n \otimes {}^0H_m & \longrightarrow & {}^0H_{n+m} \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{End}(E^n) \otimes \mathrm{End}(E^m) & \xrightarrow{\otimes} & \mathrm{End}(E^{n+m}). \end{array}$$

2.2.2. *2-representations of $\mathcal{U}^+$.*

Definition 2.3. A 2-representation of $\mathcal{U}^+$ on a category $\mathcal{V}$ is a strict monoidal functor $\mathcal{U}^+ \to \mathrm{End}(\mathcal{V})$. The data of such a functor consists of an endofunctor E of $\mathcal{V}$ and natural transformations $x \in \mathrm{End}(E)$, $\tau \in \mathrm{End}(E^2)$ satisfying (2.1)–(2.3). A morphism of 2-representations $(\mathcal{V}, E, x, \tau) \to (\mathcal{V}', E', x', \tau')$ consists of a functor $\Phi : \mathcal{V} \to \mathcal{V}'$ and an isomorphism of functors $\varphi : \Phi E \xrightarrow{\sim} E'\Phi$ such that:

$$\varphi \circ \Phi x = x'\Phi \circ \varphi : \Phi E \to E'\Phi,$$
$$E'\varphi \circ \varphi E \circ \Phi\tau = \tau'\Phi \circ E'\varphi \circ \varphi E : \Phi E^2 \to E'^2\Phi.$$

Note that $\mathrm{End}(\mathcal{V})$ is the full sub-2-category of the 2-category of categories Cat generated by the object $\mathcal{V}$. One can define $\mathcal{U}^+$ as a 2-category with a single object, so that the data of 2-representation is the data of 2-functor from $\mathcal{U}^+$ to Cat. This justifies our '2' prefixes.

In this paper we study monoidal functors from $\mathcal{U}^+$ to monoidal categories of the form $\mathsf{Bim}_k(A)$ which are defined for k-algebras A as follows: the objects of $\mathsf{Bim}_k(A)$ are (A, A)-bimodules, and the morphisms of $\mathsf{Bim}_k(A)$ are bimodule maps. The monoidal structure on $\mathsf{Bim}_k(A)$ is given by tensor product of bimodules over A.

Note that there is a 2-category Alg_k with k-algebras, bimodules, and bimodule maps as the objects, 1-morphisms, and 2-morphisms. Then $\mathsf{Bim}_k(A)$ is the full sub-2-category of Alg_k generated by the object A.

Proposition 2.4. *The data of a 2-representation $\mathcal{U}^+ \to \mathsf{Bim}_k(A)$ for a k-algebra A consists of a bimodule ${}_AE_A$ and bimodule maps $x \in \operatorname{End}(E)$, $\tau \in \operatorname{End}(E^2)$ that satisfy (strictly) the relations of $\mathcal{U}^+$.*

We will use 'x_i' and 'τ_i' to denote the generators in any ${}^0H_n$ (where $i \leqslant n$ for x_i and $i < n$ for τ_i are assumed). Given a 2-representation for a k-algebra A with bimodule E, these symbols are also used to denote the corresponding elements in each $\operatorname{End}(E^n)$.

2.2.3. *The 2-representation $\mathcal{L}(1)$.* A simple 2-representation of $\mathcal{U}^+$ is given for the algebra $A = A_{+1} \times A_{-1}$, $A_i = k[y]$, by the bimodule $E = k[y]$, where $y \in A_{-1}$ acts on the left by 0 and on the right by multiplication by y, and $y \in A_{+1}$ acts on the right by 0 and the left by y. The Hecke actions are generated by $x \in \operatorname{End}(E)$ acting by multiplication by y, and $\tau \in \operatorname{End}(E^2)$ satisfies $\tau = 0$ because $E^2 = 0$.

2.3. **Further conventions.** Assume we are given data (A, E, x, τ) determining a 2-representation, and fix these through §4. Assume that ${}_AE$ is f.g. projective and that E^n is free as a P_n-module.

Consider the endomorphism $x-y$ of the $(A[y], A[y])$-bimodule $E[y]$. Its image $(x-y)E[y]$ is a sub-bimodule of $E[y]$. Write E_y for the quotient $E[y]/(x-y)E[y]$. (Alternatively: E_y is E extended to an $(A[y], A[y])$-bimodule by specifying that y acts on both sides by x.) The projection

$$\begin{aligned} \pi : E[y] &\to E_y \\ ey^n &\mapsto x^n(e) \end{aligned}$$

is a surjection of bimodules.

We simplify notation for tensor products by adopting a convention that concatenation indicates the tensor product over an algebra that is clear from the context. Sometimes it will be unclear whether a tensor product is meant over A or over $A[y]$, so we further stipulate that if the expression for a module contains 'y', it will be understood as an $A[y]$-module, and if the expression lacks 'y', it will be understood as an A-module. Concatenation will indicate tensor product over $A[y]$ if both are $A[y]$-modules, otherwise it will indicate tensor product over A.

We will tacitly use canonical isomorphisms such as

$$M[y] \otimes_{A[y]} N[y] \xrightarrow{\sim} M[y] \otimes_A N \xrightarrow{\sim} (MN)[y]$$

for M a right A-module and N a left A-module. For example, EE_y denotes $E \otimes_A E_y$ according to our convention, but this is canonically isomorphic to $E[y] \otimes_{A[y]} E_y$, and the latter may be written $E[y]E_y$. So we may write either EE_y or $E[y]E_y$ with equivalent meanings.

Extend x to $\operatorname{End}(E[y])$ by $x : ey^n \mapsto x(e)y^n$ and τ to $\operatorname{End}(E^2[y])$ by $\tau : eey^n \mapsto \tau(ee)y^n$. The map s defined above in terms of x and τ extends in the same manner to a map in $\operatorname{End}(E^2[y])$. Note that we denote an arbitrary element of $E[y]$ by the single letter 'e'. Similarly an arbitrary element of $E^2[y]$ is denoted by the doubled symbol 'ee', which may well not be a simple tensor of the form $e \otimes e$. Later we will use 'eee' or 'eee_i' as suggestive notation for elements of $E^3[y]$, and so on.

Define $\delta = \tau \circ (Ex - y) \in \mathrm{End}(E^2[y])$. We also consider the extensions of x_i and τ_i to $E^n[y]$, and then s_i and δ_i defined by their same formulas but replacing x with x_i and τ with τ_i. Some important identities are quickly verified:

Lemma 2.5. *We have*

- $s^2 = 1$*, so* s *is an isomorphism*
- $\delta^2 = \delta$*, so* δ *is an idempotent,*

and we also have $s_i^2 = 1$ *and* $\delta_i^2 = \delta_i$*.*

We adopt a flexible notation $y_i = x_i - y$ until §5. Here y_i indicates $\big(E^j x E^{i-1} - y\big)$ for some j, and context will determine the value of j. Note that $\delta_i = \tau_i y_i$.

One may check that $s \circ x_2 = x_1 \circ s$ and $s \circ x_1 = x_2 \circ s$. It follows that s exchanges y_2 and y_1 and descends to a map:

$$s : E_y \otimes_{A[y]} E[y] \to E[y] \otimes_{A[y]} E_y.$$

So we have $s : E^2 \to E^2$ a map of (A, A)-bimodules, and this induces $s : E^2[y] \to E^2[y]$ as well as $s : E_y E \to E E_y$, maps of $(A[y], A[y])$-bimodules. Context will determine the domain and codomain for the symbol s.

Lemma 2.6. *We also have:*

- $\pi_1 \circ \delta = s \circ \pi_2 : E^2[y] \to E E_y$*.*

We define projections $\pi_i : E^n[y] \to E^{n-i} E_y E^{i-1} = E^n[y]/(y_i)$ by $\pi_i = E^{n-i} \pi E^{i-1}$. The same names may be used for maps between products with E_y factors, for example $\pi_2 : E E_y \to E_y E_y$.

Given a module ${}_A M$, its algebra of endomorphisms $\mathrm{End}_A({}_A M)$ will use the traditional order of composition for multiplication: $(f \circ g)(m) = f(g(m))$. Typically, but not always, '$\circ$' is written to emphasize this convention. A consequence is that for a ring A, the algebra $\mathrm{End}_A({}_A A)$ is identified with A^{op}.

Given two complexes M, N of A-modules, we will write $\mathscr{H}om_A(M, N)$ for the complex generated by homogeneous A-module homomorphisms from M to N. In degree n it is given by homogeneous maps of degree n, and the differential is $d(f) = d \circ f - (-1)^{|f|} f \circ d$ for f a homogeneous map of degree $|f|$. The notation $Z^i M$ refers to the degree i part of the kernel of d.

Given an algebra R, we write $D^b(R)$ for the derived category of bounded complexes of left R-modules. A strictly perfect complex of left R-modules is a bounded complex of finitely generated projective R-modules. The category per $R \subset D^b(R)$ is the full subcategory of complexes quasi-isomorphic to strictly perfect complexes. Given $M \in D^b(R)$, we write $\langle M \rangle_\Delta$ for the smallest triangulated strictly full subcategory of $D^b(R)$ closed under direct summands and containing M.

Lemma 2.7. *We have* $\langle R \rangle_\Delta = per\ R$*.*

2.4. **Generalized matrix algebras and tensor product.** Suppose we are given k-algebras A and D, bimodules ${}_AB_D$ and ${}_DC_A$, and bimodule maps

$$
{}_AB \otimes_D C_A \xrightarrow{\gamma_1} A
$$
$$
{}_DC \otimes_A B_D \xrightarrow{\gamma_2} D.
$$

With this data we can define a new k-algebra R:

$$
R = \begin{pmatrix} A & B \\ C & D \end{pmatrix},
$$

where multiplication of matrices is defined with the customary formulas using the above bimodule structures and maps.

A right R-module consists of the data of M_1 a right A-module, M_2 a right D-module, a map $M_1 \otimes_A B \xrightarrow{\alpha} M_2$ of right D-modules, and a map $M_2 \otimes_D C \xrightarrow{\beta} M_1$ of right A-modules, such that the latter two maps are compatible with γ_1 and γ_2. Here compatibility with γ_1, for example, means that the following compositions agree:

$$
M_1 \otimes_A (B \otimes_D C) \xrightarrow{\mathrm{Id}_{M_1} \otimes \gamma_1} M_1 \otimes_A A \xrightarrow{\sim} M_1
$$
$$
(M_1 \otimes_A B) \otimes_D C \xrightarrow{\alpha \otimes \mathrm{Id}_C} M_2 \otimes_D C \xrightarrow{\beta} M_1.
$$

The data of a left R-module may be given in a similar form.

Let

$$
M = \begin{pmatrix} M_1 & M_2 \end{pmatrix}
$$

be a right R-module, and

$$
N = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}
$$

a left R-module. Their tensor product $M \otimes_R N$ may be formed as follows. Consider the pair of maps given by the R action data:

$$
M_1 \otimes_A B \otimes_D N_2 \xrightarrow{I_B} M_1 \otimes_A N_1 \oplus M_2 \otimes_D N_2
$$
$$
M_2 \otimes_D C \otimes_A N_1 \xrightarrow{I_C} M_1 \otimes_A N_1 \oplus M_2 \otimes_D N_2
$$

by $I_B(m \otimes b \otimes n) = m \otimes b.n - m.b \otimes n$ and likewise for I_C. Then we have an isomorphism:

$$
(M_1 \otimes_A N_1 \oplus M_2 \otimes_D N_2) \big/ (I_B + I_C) \xrightarrow{\sim} M \otimes_R N.
$$

Now let $F \in \mathrm{End}_R(N)$ be an endomorphism of left R-modules. It determines an endomorphism $\mathrm{Id}_M \otimes_R F \in \mathrm{End}_k(M \otimes_R N)$ which will be denoted MF. We can study this on components as follows. There are induced endomorphisms $F_1 \in \mathrm{End}_A(N_1)$ and $F_2 \in \mathrm{End}_D(N_2)$ given by restriction of F. These determine endomorphisms $M_1F_1 \in \mathrm{End}_k(M_1 \otimes_A N_1)$ and $M_2F_2 \in \mathrm{End}_k(M_2 \otimes_D N_2)$, and these in turn provide together an endomorphism $\begin{pmatrix} M_1F_1 & 0 \\ 0 & M_2F_2 \end{pmatrix}$ of $M_1 \otimes_A N_1 \oplus M_2 \otimes_D N_2$. The property of full R-linearity of F implies that this morphism preserves the submodules I_B and I_C, and descends to the quotient $M \otimes_R N$ where it agrees with MF.

Lemma 2.8. *In the notations used above, an element of* $\mathrm{End}_k(M \otimes_R N)$ *of the form* MF *for* $F \in \mathrm{End}_R(N)$ *is uniquely determined by the induced maps* M_1F_1 *and* M_2F_2*.*

3. Product category

Given a 2-representation $\mathcal{V}$ for A with $\mathcal{U}^+$-action data (E, x, τ), we seek a 2-representation for C with data $(\tilde{E}, \tilde{x}, \tilde{\tau})$ to serve as the tensor 2-product $\mathcal{L}(1) \circledast \mathcal{V}$. In this section we describe our proposal for the algebra C and data $(\tilde{E}, \tilde{x}, \tilde{\tau})$, and in the next section we study this data and verify that the nil affine Hecke relations hold for $\tilde{x}$ and $\tilde{\tau}$.

3.1. Naive product category.

3.1.1. *Naive product algebra B.*

Definition 3.1. Let B be the k-algebra:

$$B = \begin{pmatrix} A[y] & E_y \\ 0 & A[y] \end{pmatrix}.$$

Here the algebra structure of B is given by matrix multiplication, with the $(A[y], A[y])$-bimodule structure of E_y contributing for products with generators in B_{12}.

A left B-module consists of a pair $\left(\begin{smallmatrix} M_1 \\ M_2 \end{smallmatrix}\right)$ of left $A[y]$-modules, together with a morphism $\alpha : E_y \otimes_{A[y]} M_2 \to M_1$ of left $A[y]$-modules. A right B-module is the data of a pair $\left(\begin{smallmatrix} N_1 & N_2 \end{smallmatrix}\right)$ of right $A[y]$-modules, together with a morphism $\beta : N_1 \otimes_{A[y]} E_y \to N_2$ of right $A[y]$-modules. It follows that a (B, B)-bimodule can be written as a matrix of $(A[y], A[y])$-bimodules with accompanying maps α and β giving left and right actions of E_y. Such a matrix with α, β determines a (B, B)-bimodule only if the actions commute. Usually this commutativity is obvious and we do not bother to check it.

A complex of left B-modules is the same data as a pair of complexes of $A[y]$-modules together with a morphism α of complexes; note that the differential of $E_y \otimes M_2$ for a complex (M_2, d) is just $E_y \otimes d$. Similarly for right B-module complexes.

3.1.2. *Endofunctor E' of B-cplx.*

Definition 3.2. Let E' be the following bounded complex of (B, B)-bimodules concentrated in degrees 0 and 1:

$$E' = \begin{pmatrix} E[y] & E[y]E_y \\ 0 & E[y] \end{pmatrix} \xrightarrow{d} \begin{pmatrix} E_y & E_yE_y \\ A[y] & E_y \end{pmatrix}.$$

Here the left action data 'α' for B generators in E_y is given on the degree 0 part as a matrix using the decompositions $0 \oplus E_yE[y]$ and $E[y] \oplus E[y]E_y$ by $\left(\begin{smallmatrix} 0 & 0 \\ 0 & s \end{smallmatrix}\right)$, and on the degree 1 part by $\begin{pmatrix} \mathrm{Id}_{E_y} & 0 \\ 0 & \mathrm{Id}_{E_yE_y} \end{pmatrix}$. The right action on the degree 0 part is given by $\begin{pmatrix} \mathrm{Id}_{E[y]E_y} & 0 \\ 0 & 0 \end{pmatrix}$ and on degree 1 it is given by $\begin{pmatrix} \mathrm{Id}_{E_yE_y} & 0 \\ 0 & \mathrm{Id}_{E_y} \end{pmatrix}$. The differential d is given componentwise by $\left(\begin{smallmatrix} \pi & \pi \otimes \mathrm{Id}_{E_y} \\ 0 & \pi \end{smallmatrix}\right)$.

Tensoring by E' on the left gives an endofunctor ${}_BE' \otimes_B -$ of the category of complexes of B-modules. It is convenient to have a formula for the action of this endofunctor on an arbitrary complex of modules:

Lemma 3.3. *Let $M = \left(\left(\begin{smallmatrix} M_1 \\ M_2 \end{smallmatrix}\right), \alpha\right)$ be a complex of B-modules. The action of the functor $E' \otimes_B -$ on M is given by:*

$$\left(\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}, \alpha\right) \xmapsto{E'} \left(\begin{pmatrix} E[y]M_1 \overset{\pi M_1}{\overset{\curvearrowright}{\oplus}} E_yM_1[-1] \\ E[y]M_2 \overset{\alpha\circ\pi M_2}{\overset{\curvearrowright}{\oplus}} M_1[-1] \end{pmatrix}, \begin{pmatrix} E[y]\alpha \circ sM_2 & 0 \\ 0 & Id_{E_yM_1} \end{pmatrix}\right).$$

Here the top and bottom rows express cocones of the maps πM_1 and $\alpha \circ \pi M_2$.

Remark 3.4. It may help motivation to consider the effect of E' at the level of the Grothendieck group when M_1 and M_2 are just modules, not complexes. The following discussion is not intended to be precise or complete.

Suppose M_1' and M_2' are projective left A-modules, and R_1 and R_2 are projective left $k[y]$-modules. Consider the projective left $A[y]$-modules $M_1 = R_1 \otimes_k M_1'$ and $M_2 = R_2 \otimes_k M_2'$. These are elements of the outer product of categories $(k[y]\text{-proj}) \boxtimes_k (A\text{-proj})$. Suppose $\alpha : E_yM_2 \to M_1$ is given. Apply E' to $\left(\left(\begin{smallmatrix} M_1 \\ M_2 \end{smallmatrix}\right), \alpha\right)$. The upper row is quasi-isomorphic to:

$$\ker\left(E[y]M_1 \xrightarrow{\pi M_1} E_yM_1\right) \xrightarrow{\sim} (y_1E[y])M_1 \xrightarrow{\sim} E[y]M_1 \xrightarrow{\sim} R_1 \otimes_k (E \otimes_A M_1'),$$

where the first isomorphism follows by flatness of M_1. Letting e denote the action of E on the Grothendieck group, we have $(1 \otimes e)\left([R_1] \otimes_k [M_1']\right)$ for the upper row in the Grothendieck group. The lower row is the cocone of α, which contributes $[E[y]M_2] + [M_1]$ in the Grothendieck group. Now recall that the raising functor for $\mathcal{L}(1)$ is just $k[y]$. So:

$$M_1 \xrightarrow{\sim} (k[y] \otimes 1)\left(R_1 \otimes_k M_1'\right), \quad [M_1] = (e \otimes 1)\left([R_1] \otimes_k [M_1']\right),$$

and we should interpret the copy of M_1 coming from the lower row in this way, since the factor of $k[y]$ in the $A[y] \cong k[y] \otimes_k A$ of the lower left corner of B is the higher weight copy. We also have $[E[y]M_2] = (e \otimes 1)\left([R_2] \otimes_k [M_2']\right)$. Finally, it is a fact that $(e \otimes 1)\left([R_2] \otimes_k [M_2']\right) = 0$ because $\mathcal{L}(1)$ has only two weight categories. It follows from these calculations that the action of $e' = [E']$ on the Grothendieck group of the derived category has the form:

$$\begin{aligned} e'[\left(\left(\begin{smallmatrix} M_1 \\ M_2 \end{smallmatrix}\right), \alpha\right)] &:= [E'\left(\left(\begin{smallmatrix} M_1 \\ M_2 \end{smallmatrix}\right), \alpha\right)] \\ &= (e \otimes 1 + 1 \otimes e)\left([M_1'] \otimes_k [R_1] + [M_2'] \otimes_k [R_2]\right). \end{aligned}$$

This agrees with the Hopf coproduct formula $\Delta(e) = e \otimes 1 + 1 \otimes e$.

Proof of the lemma. We first check that the matrix specifying the new E_y action gives a morphism of complexes. The diagonal coefficients of the matrix give morphisms of the separate summands, and these commute with the differentials on the separate summands. It remains to see that $\pi M_1 \circ E[y]\alpha \circ sM_2 = \mathrm{Id}_{E_y} M_1 \circ E_y(\alpha \circ \pi M_2)$, and these agree because $\pi E_y \circ s = E_y\pi$.

Now we compute the tensor product following the recipe of §2.4. We have:

$$E' \otimes_B M = \begin{pmatrix} \Big(E[y]M_1 \oplus E[y]E_y M_2\Big)/I_1 \overset{\pi M_1}{\overset{\frown}{\oplus}} \Big(\Big(E_y M_1 \oplus E_y E_y M_2\Big)/I_1'\Big)[-1] \\ \Big(0 \oplus E[y]M_2\Big)/I_2 \overset{\alpha\circ\pi M_2}{\overset{\frown}{\oplus}} \Big(\Big(A[y]M_1 \oplus E_y M_2\Big)/I_2'\Big)[-1] \end{pmatrix}.$$

Here the submodule I_1 is generated by all terms of the form $e \otimes \alpha(e', m_2) - e \otimes e' \otimes m_2$ for $e \in E[y], e' \in E_y, m_2 \in M_2$. So every element of the quotient has a canonical representative in $E[y]M_1$, and the quotient is isomorphic to $E[y]M_1$. With analogous reasoning we see that the quotient by I_1' is isomorphic to $E_y M_1$, that by I_2 is isomorphic to $E[y]M_2$, and that by I_2' is isomorphic to M_1. The differential may be written before taking quotients as dM_1 on the top and dM_2 on the bottom. The images of dM_2 in $E_y M_2$ represent elements in M_1 by way of α, and this determines the differential component $\alpha \circ \pi M_2$ between summands of the bottom row.

Now we calculate the new E_y action in order to view this as a complex of B-modules. Using the description of the left B-action on E', one sees that the action on the left summand is by sM_2, which is represented in $E[y]M_1$ through α, so the action written on the quotients as described above is given by $E[y]\alpha \circ sM_2$. The action is obvious on the right summand. □

3.1.3. *Category per B and generator X.*

Definition 3.5. Let X be the following complex of B-modules:

$$X = X_1 \oplus X_2$$
$$X_1 = \begin{pmatrix} A[y] \\ 0 \end{pmatrix}$$
$$X_2 = E'(X_1) = \begin{pmatrix} E[y] & \xrightarrow{\pi} & E_y \\ 0 & \longrightarrow & A[y] \end{pmatrix}$$

where X_1 lies in degree 0 and X_2 in degrees 0 and 1. The E_y action on X_2 is given by $E_y \otimes_{A[y]} A[y] \xrightarrow{\sim} E_y$, $e \otimes 1 \mapsto e$.

One can see that $X_1 = Be_1$ and $X_2 = E'e_1$, with $e_i \in B$ the standard matrix idempotent. Observe that there is a canonical right $A[y]$ action on Be_i and on X_i given componentwise.

Proposition 3.6. *The complex X is strictly perfect and generates per B.*

Proof. We can write X in terms of B:

$$X_1 = Be_1$$
$$X_2 = Be_1 \otimes_A E \to Be_2,$$

where the differential is by π on the upper row. This is a complex of finitely generated projective B-modules because $_AE$ is finitely generated and projective. So X is strictly perfect. To see that X generates per B, first note that $Be_1 = X_1 \in \langle X \rangle_\Delta$. Now consider $Be_1 \otimes_A E$ as a complex in degree 0. There is a map of complexes $X_2 \to Be_1 \otimes_A E$ given by the identity in degree 0 and by 0 in degree 1. Then $Be_2[-1]$ (a complex in degree 1) is quasi-isomorphic to the cocone of this map. So $Be_2 \in \langle X \rangle_\Delta$. □

Recall our notation $\pi_i = E^{n-i}\pi E^{i-1} : E^n[y] \to E^{n-i}E_yE^{i-1}$.

Lemma 3.7. *The kernel of* $\varphi : E^n[y] \xrightarrow{(\pi_i)_i} \bigoplus_{i=1}^n E^{n-i}E_yE^{i-1}$ *is* $(y_1 \dots y_n)E^n[y]$.

Proof. We have assumed that E^n is free as a P_n-module. It follows that $E^n[y]$ is free as a $P_n[y]$-module. Let $e \in \ker \varphi$. So $\pi_i(e) = 0$ and therefore $e \in y_iE^n[y]$ for each $i \in \{1, \dots, n\}$. Let B be a basis of $E^n[y]$ over $P_n[y]$. Write

$$e = y_i \sum_{j=1}^{\ell} f_j^i(x_1, \dots, x_n, y) \cdot b_j$$

for $b_j \in B$ distinct and $f_j^i \in P_n[y]$. It follows that $y_if_j^i = y_kf_j^k$ in $P_n[y]$ for each $(i,k) \in \{1, \dots, n\}^{\times 2}$ and $j \in \{1, \dots, \ell\}$. Then $e = y_1 \dots y_n e^\circ$ for some $e^\circ \in E^n[y]$ because $P_n[y]$ is a unique factorization domain and each y_i is irreducible. □

Lemma 3.8. *The complex* $E'X_2$ *is concentrated in degrees* 0, 1, *and* 2*:*

$$E'X_2 = \left(\begin{pmatrix} E^2[y] \xrightarrow{(\pi_2,\pi_1)} E_yE \oplus EE_y \xrightarrow{(-\pi_1,\pi_2)} E_yE_y \\ 0 \longrightarrow E[y] \oplus E[y] \xrightarrow{(-\pi,\pi)} E_y \end{pmatrix}, \alpha \right),$$

where

$$\alpha_0 = 0$$
$$\alpha_1 = \begin{pmatrix} Id_{E_yE} & 0 \\ 0 & s \end{pmatrix}$$
$$\alpha_2 = Id_{E_yE_y}.$$

Proof. Computation. The minus signs arise from shifting differentials. □

Proposition 3.9. *The complex* $E'X$ *is quasi-isomorphic to a finite direct sum of summands of* X.

We define two complexes of B-modules before proving the proposition.

Definition 3.10. Let $R, X_2' \in B$-cplx be given by

$$R = \begin{pmatrix} E^2[y] \xrightarrow{\left(\begin{smallmatrix} \pi_2 \\ \pi_2 \circ \tau \end{smallmatrix}\right)} E_yE \oplus E_yE \\ 0 \to E[y] \oplus E[y] \end{pmatrix},$$

$$X_2' = \begin{pmatrix} \tau y_1 E^2[y] & \xrightarrow{\pi_2} & E_yE \\ 0 & \longrightarrow & E[y] \end{pmatrix},$$

both lying in degrees 0 and 1, and the E_y action on R is by the canonical map

$$E_y \otimes (E[y] \oplus E[y]) \to E_yE \oplus E_yE,$$

and on X_2' by the canonical map $E_y \otimes E[y] \to E_yE$.

Lemma 3.11. *We have that* X_2' *is a finite direct sum of summands of* X_2, *and hence of* X.

Proof. Observe first that $X_2 \otimes_A E$ is a finite direct sum of summands of X because ${}_AE$ is finitely generated projective. (Here we use the componentwise right A-action on X_2.) Using the formulas

$$\pi_2 \circ \delta = \pi_2,$$
$$\pi_2 \circ (1-\delta) = 0,$$

and $\delta \cdot (1-\delta) = 0$, one has the decomposition of $X_2 \otimes_A E$:

$$X_2 \otimes_A E = \begin{pmatrix} E^2[y] & \xrightarrow{\pi_2} & E_yE \\ 0 & \longrightarrow & E[y] \end{pmatrix}$$
$$= \begin{pmatrix} \delta \cdot E^2[y] & \xrightarrow{\pi_2} & E_yE \\ 0 & \longrightarrow & E[y] \end{pmatrix} \oplus \begin{pmatrix} (1-\delta)\cdot E^2[y] \\ 0 \end{pmatrix}.$$

□

The matrix algebra structure of the nil-affine Hecke algebra gives the following isomorphism of left $A[y]$-modules:

$$E^2[y] \xrightarrow[\left(\begin{smallmatrix}\tau y_1 \\ \tau\end{smallmatrix}\right)]{\sim} \tau y_1 E^2[y] \oplus \tau y_1 E^2[y].$$

Lemma 3.12. *There is an isomorphism $R \xrightarrow{\sim} X_2' \oplus X_2'$ in B-cplx given by the above isomorphism on the degree 0 term of the upper row, and the identity on all other terms. So R is a finite direct sum of summands of X_2, and hence of X. In particular, R is strictly perfect.*

Lemma 3.13. *There is a quasi-isomorphism $R \xrightarrow{q.i.} E'X_2$ determined by $Id_{E^2[y]}$ on the degree 0 term of the upper row and $\left(\begin{smallmatrix}1 & 0 \\ 1 & -y_1\end{smallmatrix}\right)$ on the degree 1 term of the lower row.*

Proof. We first check that the map is a morphism in B-cplx. The matrix of the morphism on the degree 1 part of the upper row, as determined by equivariance over generators of B in E_y, is given by $\left(\begin{smallmatrix}\mathrm{Id} & 0 \\ s & s\circ(x_2-x_1)\end{smallmatrix}\right)$. Observe that:

$$\begin{aligned}
&\mathrm{Id}\circ \pi_2 + 0 \circ \pi_2 \circ \tau = \pi_2;\\
&s\circ \pi_2 + s\circ (x_2-x_1)\circ \pi_2 \circ \tau\\
&\ = \pi_1 \circ s + (x_1-x_2)\circ s \circ \pi_2 \circ \tau\\
&\ = \pi_1 \circ s + \pi_1 \circ (x_1-x_2)\circ s\circ\tau\\
&\ = \pi_1 \circ \Big((x_2-x_1)\circ\tau + \mathrm{Id}\\
&\qquad + (x_1-x_2)\circ\big((x_2-x_1)\circ\tau+\mathrm{Id}\big)\circ\tau\Big) = \pi_1.
\end{aligned}$$

This shows compatibility with the differential from degree 0 in the upper row. The other compatibility checks are easier.

Now we show that the map is a quasi-isomorphism. The lower row of $E'X_2$ has H^1 given by:

$$\{(e_1,e_2)\in E[y]^{\oplus 2} \mid e_1 - e_2 = y_1 e \text{ for some } e \in E[y]\}.$$

This is also the image of the (injective) map from R in degree 1 of the lower row. The upper row of $E'X_2$ has $H^0 = \ker(d^0) = y_1y_2E^2[y]$ by Lemma 3.7. The cohomology of the upper row of R is computed as follows. We have an isomorphism:

$$E^2[y] \xrightarrow{\sim} \tau y_1 E^2[y] \oplus -y_2\tau E^2[y].$$

Notice that $\pi_2 \circ \tau$ vanishes on the first summand, and π_2 vanishes on the second. Then one may compute:

$$\ker\big(\tau y_1 E^2[y] \xrightarrow{\pi_2} E_yE\big) = \tau y_1y_2E^2[y] \subset y_1y_2E^2[y]$$

and

$$\ker\big(-y_2\tau E^2[y] \underset{\sim}{\xrightarrow{\tau}} \tau y_1E^2[y] \xrightarrow{\pi_2} E_yE\big) = -y_2\tau y_1y_2E^2[y] \subset y_1y_2E^2[y].$$

So

$$\ker\big(\left(\begin{smallmatrix}\pi_2\\ \pi_2\circ\tau\end{smallmatrix}\right)\big) \subset y_1y_2E^2[y].$$

The reverse inclusion is obvious, so H^0 of the upper row is $y_1y_2E^2[y]$. This shows that $\mathrm{Id}_{E^2[y]}$ induces an isomorphism on homology in degree 0 of the upper row. Using the decomposition and inspecting the maps above, we also see that d^0 on the upper row of R is surjective. Finally we consider H^1 of the upper row of $E'X_2$ and show it is zero. (Clearly the H^2 is zero.) Let $(ee_1, ee_2) \in E_yE \oplus EE_y$ be in $\ker(d^1)$, i.e. such that $\pi_1(ee_1) = \pi_2(ee_2)$. Then $ee_1 = ee_2 + (Ex - xE)ee^\circ$ for some $ee^\circ \in E^2$. (Note that $E_yE_y \cong E^2/(Ex - xE)$ where y acts by Ex or xE.) Then consider $ee_2 + (Ex - y)ee^\circ \in E^2[y]$. The differential d^0 sends this to ee_1 in E_yE and to ee_2 in EE_y. □

Proof of Proposition 3.9. The proposition follows from the preceding three lemmas. □

Corollary 3.14. *Tensoring with ${}_BE'_B$ gives an endofunctor $E' \otimes_B -$ of per B.*

Proof. We know that $X \in \operatorname{per} B$, and it follows from Prop. 3.9 that $E' \otimes_B X \in \operatorname{per} B$. The corollary follows because X generates per B. □

Remark 3.15. We do not know that $E' \otimes_B -$ on $K^b(B)$ is exact, so we do not know that it descends to an endofunctor defined on all of $D^b(B)$.

3.2. **Bimodules G_n.** The constructions of this paper make use of certain bimodules that we describe next.

Definition 3.16. Let G_n denote $\mathrm{Hom}_{K^b(B)}(X_2, E'^nX_1)$.

Every G_n has the structure of $(G_1^{\mathsf{op}}, A[y])$-bimodule by pre- and post-composition. Here we understand $A[y] \cong \mathrm{End}_{K^b(B)}(X_1)^{\mathsf{op}}$ and use functoriality of E' for the action. Note that $G_1 = \mathrm{Hom}_{K^b(B)}(X_2, X_2)$ has an algebra structure, and the right regular action of G_1^{op} on G_1 extends the right $A[y]$ action.

In this section we gather some facts regarding these bimodules and give concrete presentations in small cases that are easier to handle. Given $n \in \{1, 2, 3, 4\}$, we define $\bar{G}_n$ as an $(A[y], A[y])$-sub-bimodule of

$$E^{n-1}[y]^{\oplus n} \oplus \mathrm{Hom}_A({}_AE, E^n)[y].$$

(By $E^0[y]$ we mean $A[y]$.) We give isomorphisms $\bar{G}_n \xrightarrow{\sim} G_n$ for such n. These isomorphisms induce left G_1^{op}-actions on $\bar{G}_n$ that extend the left $A[y]$-actions. In future sections we do not distinguish G_n from $\bar{G}_n$ and write only the former.

Definition 3.17. Define the following $(A[y], A[y])$-sub-bimodule of $A^{\mathsf{op}}[y] \oplus \mathrm{End}_A({}_AE)[y]$:

$$\begin{aligned} \bar{G}_1 = \Big\langle (\theta, \varphi) \in A^{\mathsf{op}}[y] \oplus \mathrm{End}_A({}_AE)[y] \Big| \\ \varphi = \_.\theta + y_1\varphi_1 \\ \text{for some } \varphi_1 \in \mathrm{End}_A({}_AE)[y] \Big\rangle. \end{aligned}$$

This bimodule also has a k-algebra structure with componentwise multiplication (using the opposite multiplication on generators in $A[y]$).

Note that $\bar{G}_1$ contains a copy of $A^{\mathsf{op}}[y]$, namely the subspace with $\varphi = \_.\theta$.

Proposition 3.18. *There is an isomorphism of $(A[y], A[y])$-bimodules $\bar{G}_1 \xrightarrow{\sim} G_1$ determined by:*

$$(\theta, \varphi) \mapsto \left(\begin{pmatrix} (e,0) \\ (0,1) \end{pmatrix} \mapsto \begin{pmatrix} (\varphi(e),0) \\ (0,\theta) \end{pmatrix} \right).$$

Here $(e, 0) \in E[y] \oplus E_y$ is an element of the upper row of X_2, with e in degree 0 and 0 in degree 1. Analogously with the lower row. This isomorphism respects the k-algebra structure.

Proof. The condition $\varphi = \_.\theta + y_1\varphi_1$ in the definition of $\bar{G}_1$ is equivalent to the statement that the morphism given as the image of (θ, φ) defined in the proposition has zero differential. □

Definition 3.19. Define the following $(A[y], A[y])$-sub-bimodule of $E[y]^{\oplus 2} \oplus \mathrm{Hom}_A({}_AE, E^2)[y]$:

$$\begin{aligned} \bar{G}_2 = \Big\langle (e_1, e_2, \xi) \in E[y]^{\oplus 2} \oplus \mathrm{Hom}_A({}_AE, E^2)[y] \Big| \\ e_1 - e_2 = y_1 e' \\ \xi = \_ \otimes e_1 + y_2\xi_1 \\ = \delta(\_ \otimes e_2) + y_1\xi_2 \\ \text{for some } e' \in E[y] \text{ and } \xi_\ell \in \mathrm{Hom}_A({}_AE, E^2)[y] \Big\rangle. \end{aligned}$$

Proposition 3.20. *There is an isomorphism of $(A[y], A[y])$-bimodules $\bar{G}_2 \xrightarrow{\sim} G_2$ determined by:*

$$(e_1, e_2, \xi) \mapsto \left(\begin{pmatrix} (e,0) \\ (0,1) \end{pmatrix} \mapsto \begin{pmatrix} (\xi(e),0,0) \\ (0, \left(\begin{smallmatrix} e_1 \\ e_2 \end{smallmatrix}\right), 0) \end{pmatrix} \right).$$

Proof. Use the description of $E'X_2$ in Lemma 3.8. As in Prop. 3.18, the condition of the definition of $\bar{G}_2$ is equivalent to the statement that the image of (e_1, e_2, ξ) has zero differential. □

In order to parametrize G_3, we compute the components of $E'^2X_2 = E'^3X_1$ in degrees 0, 1, and 2:

$$\begin{pmatrix} E^3[y] & \to & E_yEE \oplus EE_yE \oplus EEE_y & \to & E_yE_yE \oplus E_yEE_y \oplus EE_yE_y \to \dots \\ 0 & \to & E^2[y] \oplus E^2[y] \oplus E^2[y] & \to & E_yE \oplus EE_y \oplus EE_y \to \dots \end{pmatrix}.$$

The upper left differential map is (π_3, π_2, π_1). We don't make use of the upper right. The bottom right differential map is given by the matrix:

$$\begin{pmatrix} -\pi_2 & \pi_2 & 0 \\ -\pi_1 & 0 & \pi_1 \circ \delta \\ 0 & -\pi_1 & \pi_1 \end{pmatrix}.$$

Definition 3.21. Define the following $(A[y], A[y])$-sub-bimodule of $E^2[y]^{\oplus 3} \oplus \mathrm{Hom}_A({}_AE, E^3)[y]$:

$$\begin{aligned} \bar{G}_3 = \Big\langle (ee_1, ee_2, ee_3, \chi) \in E^2[y]^{\oplus 3} \oplus \mathrm{Hom}_A({}_AE, E^3)[y] \Big| \\ ee_1 - ee_2 &= y_2 ee' \\ ee_3 - ee_2 &= y_1 ee'' \\ \delta(ee_3) - ee_1 &= y_1 ee''', \\ \chi &= \_ \otimes ee_1 + y_3\chi_1 \\ &= \delta E(\_ \otimes ee_2) + y_2\chi_2 \\ &= E\delta \circ \delta E(\_ \otimes ee_3) + y_1\chi_3 \\ &\text{for some } ee^k \in E^2[y] \text{ and } \chi_\ell \in \mathrm{Hom}_A({}_AE, E^3)[y] \Big\rangle. \end{aligned}$$

Proposition 3.22. *There is an isomorphism of $(A[y], A[y])$-bimodules $\bar{G}_3 \xrightarrow{\sim} G_3$ determined by:*

$$(ee_1, ee_2, ee_3, \chi) \mapsto \left(\begin{pmatrix} (e, 0) \\ (0, 1) \end{pmatrix} \mapsto \begin{pmatrix} (\chi(e), 0, \dots) \\ (0, \begin{pmatrix} ee_1 \\ ee_2 \\ ee_3 \end{pmatrix}, \dots) \end{pmatrix} \right).$$

Proof. The condition of the definition of $\bar{G}_3$ is equivalent to the statement that the image of (ee_1, ee_2, ee_3, χ) has zero differential. □

Definition 3.23. Define the following $(A[y], A[y])$-sub-bimodule of $E^3[y]^{\oplus 4} \oplus \mathrm{Hom}_A({}_AE, E^4)[y]$:

$$\bar{G}_4 = \Big\langle (eee_1, eee_2, eee_3, eee_4, \psi) \in E^3[y]^{\oplus 4} \oplus \mathrm{Hom}_A({}_AE, E^4)[y] \Big|$$

$$\begin{aligned}
eee_3 - eee_4 &= y_1 eee^{(1)} \\
eee_2 - eee_3 &= y_2 eee^{(2)} \\
E\delta(eee_4) - eee_2 &= y_1 eee^{(3)} \\
eee_1 - eee_2 &= y_3 eee^{(4)} \\
eee_1 - \delta E(eee_3) &= y_2 eee^{(5)} \\
eee_1 - \delta E \circ E\delta(eee_1) &= y_1 eee^{(6)} \\
\psi &= _ \otimes eee_1 + y_4 \psi_1 \\
&= \delta E^2(_ \otimes eee_2) + y_3 \psi_2 \\
&= E\delta E \circ \delta E^2(_ \otimes eee_3) + y_2 \chi_3 \\
&= E^2\delta \circ E\delta E \circ \delta E^2(_ \otimes eee_4) + y_1 \chi_4
\end{aligned}$$

$$\text{for some } eee^k \in E^3[y] \text{ and } \psi_\ell \in \mathrm{Hom}_A({}_AE, E^4)[y] \Big\rangle.$$

Lemma 3.24. *Under the conditions on eee_i in the definition, there is a unique $\overline{eee} \in E^3[y]$ such that:*

$$eee^{(5)} - eee^{(2)} = y_3 \overline{eee},$$
$$eee^{(4)} - \tau E(eee_3) = y_2 \overline{eee}.$$

Proof. Subtracting two equations from those conditions:

$$\begin{aligned}
y_2 \left(eee^{(5)} - eee^{(2)}\right) &= eee_1 - eee_2 - y_3 \tau E(eee_3) \\
&= y_3 \left(eee^{(4)} - \tau E(eee_3)\right)
\end{aligned}$$

By Lemma 3.7 we know there is some $\overline{eee}$ satisfying the claim. It is unique because the y_i are injective. ☐

Proposition 3.25. *There is an isomorphism of $(A[y], A[y])$-bimodules $\bar{G}_4 \xrightarrow{\sim} G_4$ determined by:*

$$(eee_1, eee_2, eee_3, eee_4, \psi) \mapsto \left(\begin{pmatrix} (e,0) \\ (0,1) \end{pmatrix} \mapsto \begin{pmatrix} (\psi(e), 0, \dots) \\ (0, \begin{pmatrix} eee_1 \\ eee_2 \\ eee_3 \\ eee_4 \end{pmatrix}, \dots) \end{pmatrix} \right).$$

Proof. The reader may compute the first terms of $E'^4 X_1$ and show that the condition of the definition of $\bar{G}_4$ is equivalent to the statement that the image of $(ee_1, ee_2, ee_3, ee_4, \psi)$ defined in the proposition has zero differential. There is some ambiguity in the order of summands in degree 1 of the lower row. The convention we have used is that the first summand arises from the latest application of E' which moves a term from degree 0 of the upper row to degree 1 of the lower (and increments the exponents on existing terms in the lower row). ☐

It will be useful to describe alternative, equivalent, conditions defining $\bar{G}_2$ and $\bar{G}_3$. It is sometimes easier to work with them.

Proposition 3.26. *Given $(e_1, e_2, \xi) \in E[y]^{\oplus 2} \oplus \mathrm{Hom}_A({}_AE, E^2)[y]$ with $e_1 - e_2 = y_1 e'$, the following conditions are equivalent:*

$$\begin{aligned} \xi &= \_ \otimes e_1 + y_2 \xi_1 \\ &= \delta(\_ \otimes e_2) + y_1 \xi_2 \\ &\quad \textit{for some } \xi_\ell \in \mathrm{Hom}_A({}_AE, E^2)[y] \end{aligned}$$

and

$$\begin{aligned} \xi &= \_ \otimes e_1 + y_2 \xi_1 \\ \xi_1 &= \tau(\_ \otimes e_2) + y_1 \xi' \\ &\quad \textit{for some } \xi' \in \mathrm{Hom}_A({}_AE, E^2)[y]. \end{aligned}$$

When these conditions hold, the ξ_ℓ and ξ' are uniquely determined by the data (e_1, e_2, ξ), and $\xi_2 = \_ \otimes e' + y_2 \xi'$.

Proof. Suppose the first condition holds. Using $\delta = y_2 \tau + \mathrm{Id}$ and $e_1 - e_2 = (x - y)e'$, we can rearrange the first equality:

$$\_ \otimes e_1 + y_2 \xi_1 = y_1 \xi_2 + y_2 \tau(\_ \otimes e_2) + \_ \otimes e_2,$$

from which

$$y_2 \Big(\xi_1 - \tau(\_ \otimes e_2) \Big) = y_1 \Big(\xi_2 - \_ \otimes e' \Big).$$

By Lemma 3.7, the image of $\xi_1 - \tau(\_ \otimes e_2)$ is in $y_1 y_2 E^2[y]$. We can then make the following definition:

$$\xi' = y_1^{-1}(\xi_1 - \tau(\_ \otimes e_2)).$$

The second condition and the final claim follow from this.

Starting now with the second condition, plugging the second equation into the first, we find:

$$\begin{aligned} \xi &= \_ \otimes e_1 + y_2 \big(\tau(\_ \otimes e_2) + y_1 \xi' \big) \\ &= \delta(\_ \otimes e_2) + \_ \otimes (e_1 - e_2) + y_2 y_1 \xi' \\ &= \delta(\_ \otimes e_2) + y_1 \big(\_ \otimes e' + y_2 \xi' \big). \end{aligned}$$

This is the second line of the first condition, and it establishes the final claim.

The uniqueness claims are clear. □

Proposition 3.27. *Given $(ee_1, ee_2, ee_3, \chi) \in E^2[y]^{\oplus 3} \oplus \mathrm{Hom}_A({}_AE, E^3)[y]$ with*

$$ee_1 - ee_2 = y_2 ee' \tag{3.1}$$

$$ee_3 - ee_2 = y_1 ee'' \tag{3.2}$$

$$\delta(ee_3) - ee_1 = y_1 ee''', \tag{3.3}$$

the following conditions are equivalent:

$$\begin{aligned}\chi &= \_ \otimes ee_1 + y_3\chi_1 \\ &= \delta E(\_ \otimes ee_2) + y_2\chi_2 \\ &= E\delta \circ \delta E(\_ \otimes ee_3) + y_1\chi_3 \\ &\textit{for some } \chi_\ell \in \mathrm{Hom}_A({}_AE, E^3)[y]\end{aligned}$$

and

$$\begin{aligned}\chi &= \_ \otimes ee_1 + y_3\chi_1 \\ \chi_1 &= \tau E(\_ \otimes ee_2) + y_2\chi_1' \\ \chi_1' &= E\tau \circ \tau E(\_ \otimes ee_3) + y_1\chi'' \\ &\textit{for some } \chi'' \in \mathrm{Hom}_A({}_AE, E^3)[y].\end{aligned}$$

When the conditions hold, the χ_ℓ and χ'' are uniquely determined by the data (ee_1, ee_2, ee_3, χ), and there is a unique $\overline{ee} \in E^2[y]$ such that

$$\begin{aligned}\tau(ee_3) - ee' &= y_1\overline{ee} \\ ee''' - ee'' &= y_2\overline{ee}.\end{aligned}$$

Define a map $\chi_2' = -\_ \otimes \overline{ee} + y_3\chi''$. Then we also have

$$\chi_2 = E\tau \circ \delta E(\_ \otimes ee_3) + y_1\chi_2'$$

and

$$\chi_3 = -\delta E(\_ \otimes ee'') + y_2\chi_2'.$$

Assuming $\chi = \_ \otimes ee_1 + y_3\chi_1$, the other two conditions together are equivalent to a single condition on χ_1:

$$\chi_1 = -\tau E y_1(\_ \otimes ee'') + E\delta \circ \tau E(\_ \otimes ee_3) + y_2y_1\chi''.$$

Proof. Suppose the first condition holds. Equating the first two formulas for χ in the first condition and using $\delta E = y_3\tau E + \mathrm{Id}$ gives:

$$\_ \otimes ee_1 + y_3\chi_1 = y_3\tau E(\_ \otimes ee_2) + \_ \otimes ee_2 + y_2\chi_2$$

thus

$$y_3\big(\chi_1 - \tau E(\_ \otimes ee_2)\big) = y_2\big(\chi_2 - \_ \otimes ee'\big).$$

By Lemma 3.7 again, the image of this function lies in $y_2y_3E^3[y]$, and since each y_i is injective, we can define a new function χ_1' such that:

$$\begin{aligned}\chi_1 &= \tau E(\_ \otimes ee_2) + y_2\chi_1' \\ \chi_2 &= \_ \otimes ee' + y_3\chi_1'.\end{aligned}$$

Equating now the second and third formulas, we have:

$$y_2E\tau \circ \delta E(\_ \otimes ee_3) + \delta E(\_ \otimes ee_3) + y_1\chi_3 = \delta E(\_ \otimes ee_2) + y_2\chi_2$$

so

$$y_2\big(\chi_2 - E\tau \circ \delta E(\_ \otimes ee_3)\big) = y_1\big(\chi_3 + \delta E(\_ \otimes ee'')\big),$$

so for some χ_2' we can write:

$$\chi_2 = E\tau \circ \delta E(\_ \otimes ee_3) + y_1\chi_2'$$
$$\chi_3 = -\delta E(\_ \otimes ee'') + y_2\chi_2'.$$

We will need a fact derived from the relations (3.1)–(3.3) of the ee^k. Adding the first and third relations and subtracting the second yields

$$y_1\big(ee''' - ee''\big) = y_2\big(\tau(ee_3) - ee'\big),$$

from which we see there must be a (unique) $\overline{ee}$ with

$$\tau(ee_3) - ee' = y_1\overline{ee}$$
$$ee''' - ee'' = y_2\overline{ee}.$$

This gives the third claim of the proposition.

Equating now the two formulas we derived for χ_2:

$$y_3E\tau \circ \tau E(\_ \otimes ee_3) + E\tau(\_ \otimes ee_3) + y_1\chi_2' = \_ \otimes ee' + y_3\chi_1'$$

so

$$y_3\big(\chi_1' - E\tau \circ \tau E(\_ \otimes ee_3)\big) = y_1\big(\chi_2' + \_ \otimes \overline{ee}\big).$$

Therefore

$$\chi_1' = E\tau \circ \tau E(\_ \otimes ee_3) + y_1\chi''$$
$$\chi_2' = -\_ \otimes \overline{ee} + y_3\chi''$$

for some χ'', as desired.

In the reverse direction, starting with the second condition, plugging the χ_1 and χ_1' formulas into the first χ formula gives:

$$\chi = \_ \otimes ee_1 + y_3\Big(\tau E(\_ \otimes ee_2) + y_2\big(E\tau \circ \tau E(\_ \otimes ee_3) + y_1\chi''\big)\Big),$$

so

$$\begin{aligned}\chi - \delta E(\_ \otimes ee_2) &= \_ \otimes (ee_1 - ee_2) \\ &\qquad + y_2\big(E\tau \circ \tau E(\_ \otimes ee_3) + y_1\chi''\big) \\ &= y_2\Big(\_ \otimes ee' + E\tau \circ \tau E(\_ \otimes ee_3) + y_1\chi''\Big),\end{aligned}$$

as desired. Similarly:

$$\begin{aligned}\chi - E\delta \circ \delta E(\_ \otimes ee_3) &= \chi - y_3y_2E\tau \circ \tau E(\_ \otimes ee_3) \\ &\qquad - y_3\tau E(\_ \otimes ee_3) - E\delta(\_ \otimes ee_3) \\ &= \_ \otimes ee_1 + y_3\big(\tau E(\_ \otimes ee_2) + y_1y_2\chi''\big) \\ &\qquad - y_3\tau E(\_ \otimes ee_3) - E\delta(\_ \otimes ee_3) \\ &= \_ \otimes \big(ee_1 - \delta(ee_3)\big) + y_1\Big(-y_3\tau E(\_ \otimes ee'') + y_2y_3\chi''\Big) \\ &= y_1\Big(-\_ \otimes ee''' - y_3\tau E(\_ \otimes ee'') + y_2y_3\chi''\Big).\end{aligned}$$

The final statement of the proposition is a rearrangement of the second and third equalities of the second condition. □

Remark 3.28. We will not need to use alternative conditions for G_n for $n \geqslant 4$.

3.3. **Product category C-mod.** Let $C = \mathrm{End}_{\mathrm{per}\ B}(X)^{\mathsf{op}}$. We 'change basis' from $Be_1 \oplus Be_2$ to $X_1 \oplus X_2$, i.e. from complexes of modules over B to complexes of modules over C. This is performed by $\mathscr{H}om_B(X, -)$:

$$\mathrm{per}\ B \xrightarrow[\mathscr{H}om_B(X,-)]{\sim} \mathrm{per}\ C,$$

which is a restricted Rickard (derived Morita) equivalence. It has an inverse given by $X \otimes_C -$. Under this equivalence, the action of ${}_B E' \otimes_B -$ on per B translates to ${}_C\tilde{E} \otimes_C -$ on per C, where $\tilde{E}$ is a (C, C)-bimodule that is finitely generated and projective on the left. Our main theorem says that $\mathsf{Bim}_k(C)$ has the structure of 2-representation of $\mathcal{U}^+$ using $\tilde{E}$. In this section we describe C and the derived equivalence in more detail.

3.3.1. *New algebra C.* Let $\mathscr{C} = \mathscr{E}nd_B(X_1 \oplus X_2)^{\mathsf{op}}$ be the dg-algebra of endomorphisms of X (with left-to-right composition).

Definition 3.29. Define two $(A[y], A[y])$-bimodules:

$$G_1' = A[y] \oplus \mathrm{Hom}_{A[y]}({}_{A[y]}E[y], E[y])$$

and

$$G_1'' = \mathrm{Hom}_{A[y]}({}_{A[y]}E[y], E_y).$$

The complex $\mathscr{E}nd_B(X_2)$ is given in degrees 0 and 1 by

$$G_1' \xrightarrow{d^0} G_1''$$

where

$$d^0\big((\theta(y), \varphi)\big) = \pi \circ \varphi - \pi(-).\theta(x).$$

The direct sum decomposition $X_1 \oplus X_2$ provides a matrix presentation for $\mathscr{C}$ with $\mathscr{C}_{ij} = \mathscr{H}om_B(X_i, X_j)$.

Definition 3.30. Let F denote the (A, A)-bimodule

$$F = \mathrm{Hom}_A({}_AE, A).$$

Note the canonical isomorphism

$$\mathrm{Hom}_A({}_AE, A)[y] \xrightarrow{\sim} \mathrm{Hom}_{A[y]}({}_{A[y]}E[y], A[y])$$

that exists because ${}_AE$ is finitely generated. Since ${}_AE$ and ${}_{A[y]}E[y]$ are both finitely generated projective, we also have canonical isomorphisms of functors:

$$\mathrm{Hom}_A({}_AE, -) \xrightarrow{\sim} \mathrm{Hom}_A({}_AE, A) \otimes_A -$$

$$\mathrm{Hom}_{A[y]}({}_{A[y]}E[y], -) \xrightarrow{\sim} \mathrm{Hom}_{A[y]}({}_{A[y]}E[y], A[y]) \otimes_{A[y]} -.$$

Proposition 3.31. *The algebra $\mathscr{C}$ is isomorphic to a generalized matrix algebra of complexes concentrated in degrees* 0 *and* 1*:*

$$\begin{pmatrix} A[y] & E[y] \xrightarrow{\pi} E_y \\ F[y] & G_1'^{\mathsf{op}} \xrightarrow{d^0} G_1''^{\mathsf{op}} \end{pmatrix} \xrightarrow{\sim} \begin{pmatrix} \mathscr{C}_{11} & \mathscr{C}_{12} \\ \mathscr{C}_{21} & \mathscr{C}_{22} \end{pmatrix}.$$

The map is given on components by:

- *for* $\mathscr{C}_{11}$*:*
$$A[y] \ni a \mapsto \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} a \\ 0 \end{pmatrix} \right)$$
- *for* $\mathscr{C}_{12}$*:*
$$(E[y] \to E_y) \ni (e, e') \mapsto \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} (e, e') \\ 0 \end{pmatrix} \right)$$
- *for* $\mathscr{C}_{21}$*:*
$$F[y] \ni f \mapsto \left(\begin{pmatrix} (e, 0) \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} f(e) \\ 0 \end{pmatrix} \right)$$
- *for* $\mathscr{C}_{22}$*:*
$$\big(G_1'^{\mathsf{op}} \to G_1''^{\mathsf{op}}\big) \ni \big((\theta, \varphi'), \varphi''\big) \mapsto \left(\begin{pmatrix} (e, 0) \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} \big(\varphi'(e), (\pi \circ \varphi'')(e)\big) \\ \theta \end{pmatrix} \right).$$

Proof. Computation. □

Definition 3.32. Let C denote the k-algebra $\mathrm{End}_{K^b(B)}(X)^{\mathsf{op}}$.

Sometimes we consider C to be a dg-algebra concentrated in degree 0.

Lemma 3.33. *The projection* $Z^0(\mathscr{C}) \to H^0(\mathscr{C}) = C$ *is an isomorphism. Its inverse gives an injection* $C \hookrightarrow \mathscr{C}$ *which is a quasi-isomorphism of* dg*-algebras.*

Proof. The first claim follows because $\mathscr{C}$ lies in degrees 0 and 1. For the second claim we just need that $H^1(\mathscr{C}) = 0$. It is clear that the map $\pi : E[y] \to E_y$ is surjective. We can see that d^0 is surjective as well: since ${}_{A[y]}E[y]$ is projective, $\mathrm{Hom}_{A[y]}({}_{A[y]}E[y], -)$ is exact, so
$$\mathrm{Hom}_{A[y]}({}_{A[y]}E[y], \pi) : \mathrm{Hom}_{A[y]}({}_{A[y]}E[y], E[y]) \to \mathrm{Hom}_{A[y]}({}_{A[y]}E[y], E_y)$$
is surjective. □

The injection of the lemma gives a right action of C on X.

Lemma 3.34. *The algebra* C *is isomorphic to a generalized matrix algebra:*
$$\begin{pmatrix} A[y] & y_1 E[y] \\ F[y] & G_1^{\mathsf{op}} \end{pmatrix} \xrightarrow{\sim} \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix},$$
with component maps given by (restrictions of) those in Proposition 3.31.

Proof. We have $d^0\big((\theta, \varphi)\big) = 0$ exactly when $\varphi = \_.\theta + y_1\varphi'$ for some $\varphi' \in \mathrm{Hom}_{A[y]}({}_{A[y]}E[y], E[y])$, and it follows that the map to C_{22} is an isomorphism. □

3.3.2. *Derived equivalence.* Since X is strictly perfect, the triangulated functor
$$\mathscr{H}om_B(X, -) : K^b(B) \to K^b(C)$$
descends to the derived categories and resolutions are not needed:
$$\mathscr{H}om_B(X, -) : D^b(B) \to D^b(C).$$
Since X generates per B, it is perfect as a right $\mathscr{C}$-dg-module, and then also as a complex of C-modules because the inclusion $C \hookrightarrow \mathscr{C}$ is a quasi-isomorphism. It follows that the functor restricts to a functor
$$\mathscr{H}om_B(X, -) : \mathrm{per}\ B \to \mathrm{per}\ C,$$

and this is essentially surjective because C is in the essential image. To show that the functor is fully faithful, it is enough to check endomorphisms of X and its translates, since X generates per B. The induced map:

$$\mathrm{Hom}_{D^b(B)}(X, X[i]) \to \mathrm{Hom}_{D^b(C)}(\mathscr{E}nd_B(X), \mathscr{E}nd_B(X)[i])$$

is an isomorphism for all i: with $i = 0$ both sides are canonically isomorphic to C, and the map induces the identity on C; with $i \neq 0$ both sides are 0.

The endofunctor $E' \otimes_B -$ on per B induces an endofunctor on per C using this equivalence: first apply $X \otimes_C -$, then $E' \otimes_B -$, then $\mathscr{H}om_B(X, -)$. Since X is finitely generated and strictly perfect, this induced endofunctor is isomorphic to $\mathscr{H}om_B(X, E'X) \otimes_C -$.

Remark 3.35. In the above context a theorem of Rickard shows that $\mathscr{H}om_B(X, -) : D^b(B) \to D^b(C)$ is also an equivalence of categories. We do not know $E' \otimes_B -$ to be exact, however, so we use the restricted equivalence of perfect complexes, and the full version of Rickard's theorem is not needed.

Definition 3.36. In §3, let $\mathscr{E}$ denote the (C, C)-bimodule complex $\mathscr{H}om_B(X, E'X)$.

Then we have the following:

Proposition 3.37. *For each n, the morphism of (C, C) bimodule complexes*

$$\overbrace{\mathscr{E} \otimes_C \cdots \otimes_C \mathscr{E}}^{n\text{-times}} \to \mathscr{H}om_B(X, E'^n X)$$

given by

$$f_1 \otimes \cdots \otimes f_n \mapsto E'^{n-1}(f_n) \circ E'^{n-2}(f_{n-1}) \circ \cdots \circ f_1$$

is a quasi-isomorphism. These maps give the vertical maps in diagrams of the following form, which commute:

$$\begin{array}{ccc} \mathscr{H}om_B(X, E'X)^{\otimes n} \otimes_C \mathscr{H}om_B(X, E'X)^{\otimes m} & \longrightarrow & \mathscr{H}om_B(X, E'X)^{\otimes n+m} \\ \downarrow & & \downarrow \\ \mathscr{H}om_B(X, E'^n X) \otimes_C \mathscr{H}om_B(X, E'^m X) & \xrightarrow{f \otimes g \mapsto E'^n(g) \circ f} & \mathscr{H}om_B(X, E'^{n+m} X). \end{array}$$

Proof. All diagrams contained in the following diagram commute, up to canonical isomorphisms in per B and per C:

$$\begin{array}{ccc} \text{per } B & \underset{X \otimes_C -}{\overset{\mathscr{H}om_B(X,-)}{\rightleftarrows}} & \text{per } C \\ {\scriptstyle E' \otimes_B -}\downarrow & & \downarrow{\scriptstyle \mathscr{E} \otimes_C -} \\ \text{per } B & \underset{X \otimes_C -}{\overset{\mathscr{H}om_B(X,-)}{\rightleftarrows}} & \text{per } C \\ {\scriptstyle E' \otimes_B -}\downarrow & & \downarrow{\scriptstyle \mathscr{E} \otimes_C -} \\ \text{per } B & \underset{X \otimes_C -}{\overset{\mathscr{H}om_B(X,-)}{\rightleftarrows}} & \text{per } C. \end{array}$$

This gives the first statement of the proposition. The diagrams commute by functoriality of E'. $\square$

3.4. **New bimodule $\tilde{E}$.**

3.4.1. *Definition of $\tilde{E}$.* Now we define the lead actor of this paper.

Definition 3.38. Define a (C, C)-bimodule:

$$\tilde{E} = \mathrm{Hom}_{K^b(B)}(X, E'X),$$

with left C action given by precomposition with $\varphi \in C$, and right C action given by post-composition with $E'(\varphi)$ for $\varphi \in C$.

Lemma 3.39. *For each n, the complex $\mathscr{H}om_B(X, E'^n X)$ of (C, C)-bimodules is concentrated in nonnegative degree.*

Proof. The lower row of $E'^n X$ has components in degrees at least 1, and the upper row has components in degrees at least 0. This is shown by a simple inductive argument using the formulas for X and E' in §3.1.2. It follows that there are no nonzero morphisms in $\mathscr{H}om_B(X, E'^n X)$ of negative degree. □

Proposition 3.40. *The complex $\mathscr{E} = \mathscr{H}om_B(X, E'X)$ of (C, C)-bimodules has cohomology concentrated in degree* 0*.*

Proof. We consider separately the matrix components $\mathscr{H}om_B(X_i, E'X_j)$:

- $\mathscr{H}om_B(X_1, E'X_1)$: since $X_1 = Be_1$ this is isomorphic to $e_1 E' X_1$ which is $E[y] \xrightarrow{\pi} E_y$, and π is surjective.
- $\mathscr{H}om_B(X_1, E'X_2)$: this is isomorphic to $e_1 E'^2 X_1$, which is

$$E^2[y] \xrightarrow{\binom{\pi_2}{\pi_1}} E_y E \oplus E E_y \xrightarrow{(-\pi_1 \ \pi_2)} E_y E_y.$$

  The second map is clearly surjective. Its kernel consists of pairs $(ee_1, ee_2) \in E^2$ such that $ee_1 - ee_2 = (Ex - xE)ee^\circ$ for some $ee^\circ \in E^2$. Such a pair is the image of $ee_2 + (Ex - y)ee^\circ$ in $E^2[y]$.
- $\mathscr{H}om_B(X_2, E'X_1)$: this is isomorphic to $\mathscr{C}_{22}$, and we saw that d^0 is surjective.
- $\mathscr{H}om_B(X_2, E'X_2)$: this is isomorphic to $G_2' \xrightarrow{d^0} G_2'' \xrightarrow{d^1} G_2'''$, where

$$\begin{aligned} G_2' &= E[y]^{\oplus 2} \oplus \mathrm{Hom}_{A[y]}({}_{A[y]}E[y], E^2[y]) \\ G_2'' &= E_y \oplus \mathrm{Hom}_{A[y]}({}_{A[y]}E[y], E_y E \oplus E E_y) \\ G_2''' &= \mathrm{Hom}_{A[y]}({}_{A[y]}E[y], E_y E_y), \end{aligned}$$

  with

$$\begin{aligned} d^0 &: (e_1, e_2, \xi) \mapsto \big(\pi(e_2 - e_1), (\pi_2 \circ \xi; \pi_1 \circ \xi)\big) \\ d^1 &: \big(e, (\xi'; \xi'')\big) \mapsto -\pi_1 \circ \xi' + \pi_2 \circ \xi''. \end{aligned}$$

  It is easy to see that $H^1 = 0$ and $H^2 = 0$ by applying the exact functor $\mathrm{Hom}_{A[y]}({}_{A[y]}E[y], -)$ to the sequence considered in the second bullet.

□

Corollary 3.41. *The surjection*

$$Z^0 \mathscr{H}om_B(X, E'X) \to H^0 \mathscr{H}om_B(X, E'X) = \tilde{E}$$

is an isomorphism. Its inverse gives an injection

$$\tilde{E} \hookrightarrow \mathscr{E}$$

which is a quasi-isomorphism of complexes of (C, C)-bimodules.

Remark 3.42. Whereas E' is a complex of bimodules, $\tilde{E}$ is just a bimodule. This observation is the starting point for our construction. The basis $X_1 \oplus X_2$ is designed to be more compatible with the $\mathcal{U}^+$ action in this sense.

Lemma 3.43. *As a left C-module, $\tilde{E}$ is finitely generated and projective.*

Proof. In Prop. 3.9 we saw that $E'X$ is quasi-isomorphic to a finite direct sum of summands of X, so ${}_C\tilde{E}$ is a finite direct sum of summands of C. □

Lemma 3.44. *The map $\tilde{E}^n \to \mathscr{H}om_B(X, E'^n X)$ of complexes of (C, C)-bimodules given by*

$$f_1 \otimes \cdots \otimes f_n \mapsto E'^{n-1}(f_n) \circ E'^{n-2}(f_{n-1}) \circ \cdots \circ f_1$$

is a quasi-isomorphism.

Proof. Use a copy of the morphism

$$\tilde{E} \overset{q.i.}{\hookrightarrow} \mathscr{E}$$

from Corollary 3.41 onto each factor of the product on the left in Proposition 3.37, and the fact that $\tilde{E}$ is finitely generated and projective on the left. □

Lemma 3.45. *The maps of Lemma 3.44 induce isomorphisms of (C, C)-bimodules*

$$\tilde{E}^n \xrightarrow{\sim} \mathrm{Hom}_{K^b(B)}(X, E'^n X)$$

making the following diagrams commute:

$$\begin{array}{ccc}
\tilde{E}^n \otimes_C \tilde{E}^m & \xrightarrow{\sim} & \tilde{E}^{n+m} \\
\downarrow \sim & & \downarrow \sim \\
\mathrm{Hom}_{K(B)}(X, E'^n(X)) \otimes_C \mathrm{Hom}_{K(B)}(X, E'^m(X)) & \xrightarrow{\sim} & \mathrm{Hom}_{K(B)}(X, E'^{n+m}(X)).
\end{array}$$

Proof. By Lemma 3.44, the cohomology of $\mathscr{H}om_B(X, E'^n X)$ is concentrated in degree 0. By Lemma 3.39,

$$Z^0 \mathscr{H}om_B(X, E'^n X) = H^0 \mathscr{H}om_B(X, E'^n X).$$

So the degree 0 part of the map of Lemma 3.44 is an isomorphism from $\tilde{E}^n$ to $Z^0 \mathscr{H}om_B(X, E'^n X)$, which is $\mathrm{Hom}_{K^b(B)}(X, E'^n X)$. The diagrams commute because the morphisms are restrictions of the morphisms of Proposition 3.37. □

Definition 3.46. We let $\tilde{E}^n_{ij}$ denote $\mathrm{Hom}_{K^b(B)}(X_i, E'^n X_j)$.

Defined in this way, $\tilde{E}^n_{ij}$ lies in $\mathrm{Hom}_{K^b(B)}(X, E'^n X)$, not in $\tilde{E}^n$, but we consider it also in the latter through the isomorphism of Lemma 3.45.

3.4.2. *Some low powers of $\tilde{E}$.* The bimodule $\tilde{E}$ can be presented as a matrix with ij-component $\tilde{E}_{ij}$ given by $\mathrm{Hom}_{K^b(B)}(X_i, E'X_j)$. This component is an $(\mathrm{End}(X_i)^{\mathsf{op}}, \mathrm{End}(X_j)^{\mathsf{op}})$-bimodule. Recall that $\mathrm{End}(X_1)^{\mathsf{op}} \cong A[y]$ and $\mathrm{End}(X_2)^{\mathsf{op}} \cong G_1^{\mathsf{op}}$.

Lemma 3.47. *We have*

$$(y_1 \dots y_n)E^n[y] \xrightarrow{\sim} \mathrm{Hom}_{K^b(B)}(X_1, E'^n X_1),$$

where $y_1 \dots y_n e$ is sent to the map in $K^b(B)$ determined by:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} (y_1 \dots y_n e, 0, \dots, 0) \\ 0 \end{pmatrix}.$$

Proof. Computation. Note that $E'^n X_1$ has just one term in degree 0, which is $E^n[y]$ in the upper row. The differential of $E'^n X_1$ out of this term is the map whose kernel is computed in Lemma 3.7. ☐

Proposition 3.48. *We have:*

$$\begin{pmatrix} y_1 \dots y_n E^n[y] & y_1 \dots y_{n+1} E^{n+1}[y] \\ G_n & G_{n+1} \end{pmatrix} \xrightarrow{\sim} \begin{pmatrix} \tilde{E}^n_{11} & \tilde{E}^n_{12} \\ \tilde{E}^n_{21} & \tilde{E}^n_{22} \end{pmatrix},$$

where the maps on the upper row are from Lemma 3.47, and on the lower they are from the definition of G_n.

Together with Lemma 3.45, this gives a parametrization of $\tilde{E}^n$. We may record the matrix presentations for the first few powers:

$$\begin{pmatrix} y_1 E[y] & y_1 y_2 E^2[y] \\ G_1 & G_2 \end{pmatrix} \xrightarrow{\sim} \tilde{E},$$

$$\begin{pmatrix} y_1 y_2 E^2[y] & y_1 y_2 y_3 E^3[y] \\ G_2 & G_3 \end{pmatrix} \xrightarrow{\sim} \tilde{E}^2,$$

$$\begin{pmatrix} y_1 y_2 y_3 E^3[y] & y_1 y_2 y_3 y_4 E^4[y] \\ G_3 & G_4 \end{pmatrix} \xrightarrow{\sim} \tilde{E}^3.$$

4. Hecke action

In this section we introduce (C, C)-bimodule endomorphisms $\tilde{x}$ of $\tilde{E}$ and $\tilde{\tau}$ of $\tilde{E}^2$, and show that they satisfy the relations of $\mathcal{U}^+$.

4.1. **Definition of the action.** In §4.1.1 we give formulas for endomorphisms of the separate components of $\tilde{E}$ and $\tilde{E}^2$. A few lemmas are needed first in order to show that the formulas are well-defined on components of the form G_n, $n = 1, 2, 3$. Then in §4.1.2 we argue that these componentwise definitions jointly determine a morphism of (C, C)-bimodules.

4.1.1. *Formulas for $\tilde{x}$ and $\tilde{\tau}$.*

Lemma 4.1. *Let $(\theta, \varphi) \in G_1 \subset A^{\mathsf{op}}[y] \oplus \mathrm{Hom}_A({}_AE, E)[y]$. Then $(y\theta, x \circ \varphi) \in G_1$.*

Proof. Compute:

$$\begin{aligned} x \circ \varphi - y\theta &= x({}_{-}.\theta + y_1\varphi_1) - y\theta \\ &= y_1({}_{-}.\theta + x\varphi_1). \end{aligned}$$

□

Lemma 4.2. *Let $(e_1, e_2, \xi) \in G_2 \subset E[y]^{\oplus 2} \oplus \mathrm{Hom}_A({}_AE, E^2)[y]$. Then $(ye_1, xe_2, xE \circ \xi) \in G_2$ and $(e', e', \tau \circ \xi) \in G_2$.*

Proof. For the first claim, compute:

$$\begin{aligned} xE \circ \xi - {}_{-} \otimes ye_1 &= xE \circ ({}_{-} \otimes e_1 + y_2\xi_1) - {}_{-} \otimes ye_1 \\ &= y_2({}_{-} \otimes e_1 + xE \circ \xi_1), \end{aligned}$$

and

$$\begin{aligned} xE \circ \xi - \delta({}_{-} \otimes xe_2) &= xE \circ \big(\delta({}_{-} \otimes e_2) + y_1\xi_2\big) - \delta({}_{-} \otimes xe_2) \\ &= \delta \circ Ex({}_{-} \otimes e_2) - y_1({}_{-} \otimes e_2) \\ &\qquad + y_1 xE \circ \xi_2 - \delta({}_{-} \otimes xe_2) \\ &= y_1(-{}_{-} \otimes e_2 + xE \circ \xi_2). \end{aligned}$$

For the second claim, use the alternative characterization of G_2 as given in Prop. 3.26, and compute:

$$\begin{aligned} \tau \circ \xi &= \tau({}_{-} \otimes e_1) + \tau y_2 \xi_1 \\ &= \tau({}_{-} \otimes e_1) + y_1\tau\xi_1 - \xi_1 \\ &= \tau({}_{-} \otimes e_1) + y_1\tau y_1\xi' - \xi_1 \\ &= \tau\big({}_{-} \otimes (e_1 - e_2)\big) + y_1y_2\tau\xi' \\ &= \tau y_1({}_{-} \otimes e') + y_1y_2\tau\xi' \\ &= {}_{-} \otimes e' + y_2\big(\tau({}_{-} \otimes e') + y_1\tau\xi'\big). \end{aligned}$$

The last line has the form of an element of G_2. □

Lemma 4.3. *Let $(ee_1, ee_2, ee_3, \chi) \in G_3 \subset E^2[y]^{\oplus 3} \oplus \mathrm{Hom}_A({}_AE, E^3)[y]$. Then $(ee', ee', \tau(ee_3), \tau E \circ \chi) \in G_3$.*

Proof. We use the alternative characterization of G_3 as given in Prop. 3.27, and compute:

$$\begin{aligned}
\tau E \circ \chi &= \tau E(_ \otimes ee_1) + \tau E y_3 \chi_1 \\
&= \tau E(_ \otimes ee_1) - \chi_1 + y_2 \tau E \circ \chi_1 \\
&= \tau E(_ \otimes ee_1) - \chi_1 + y_2 \tau E y_2 \big(E\tau \circ \tau E(_ \otimes ee_3) + y_1 \chi''\big) \\
&= \tau E(_ \otimes ee_1) - \chi_1 \\
&\qquad + \big(y_2 y_3 \tau E + y_2\big) \cdot \big(E\tau \circ \tau E(_ \otimes ee_3) + y_1 \chi''\big) \\
&= \tau E(_ \otimes (ee_1 - ee_2)) \\
&\qquad + y_2 y_3 \big(\tau E \circ E\tau \circ \tau E(_ \otimes ee_3) + y_1 \tau E \circ \chi''\big) \\
&= \tau E y_2(_ \otimes ee') \\
&\qquad + y_2 y_3 \Big(E\tau \circ \tau E(_ \otimes ee') + E\delta \circ \tau E(_ \otimes \overline{ee}) + y_1 \tau E \circ \chi''\Big) \\
&= _ \otimes ee' + y_3 \cdot \\
&\qquad \Big(E\delta \circ \tau E(_ \otimes ee') + y_2\big(E\delta \circ \tau E(_ \otimes \overline{ee}) + y_1 y_2 \tau E \circ \chi''\big)\Big) \\
&= _ \otimes ee' + y_3 \cdot \\
&\qquad \Big(-\tau E y_1(_ \otimes \overline{ee}) + E\delta \circ \tau E(_ \otimes \tau(ee_3)) + y_1 y_2 \tau E \circ \chi''\Big).
\end{aligned}$$

The last line has the form of an element of G_3, namely $(ee', ee', \tau(ee_3), \tau E \circ \chi)$. ∎

The element $(ee_1, ee_2, ee_3, \chi) \in G_3$ is associated (by Prop. 3.27) with further data that has been notated ee^ℓ, $\overline{ee}$, χ_ℓ, χ_1', and χ''. We record the corresponding data associated with $(ee', ee', \tau(ee_3), \tau E \circ \chi)$ using the notation $\bar{e}$ and $\bar{\chi}$ for the new versions:

$$\begin{aligned}
\bar{e}\bar{e}' &= 0 \\
\bar{e}\bar{e}'' &= \overline{ee} \\
\bar{e}\bar{e}''' &= \overline{ee} \\
\overline{\bar{e}\bar{e}} &= 0,
\end{aligned}$$

and

$$\begin{aligned}
\bar{\chi} &= (ee', ee', \tau(ee_3), \tau E \circ \chi) \\
\bar{\chi}_1 &= -\tau E y_1(_ \otimes \overline{ee}) + E\delta \circ \tau E \circ E\tau(_ \otimes ee_3) + y_1 y_2 \tau E \circ \chi'' \\
\bar{\chi}_2 &= E\tau \circ \delta E \circ E\tau(_ \otimes ee_3) + y_1 y_3 \tau E \circ \chi'' \\
\bar{\chi}_3 &= -\delta E(_ \otimes \overline{ee}) + y_2 y_3 \tau E \circ \chi'' \\
\bar{\chi}_1' &= E\tau \circ \tau E \circ E\tau(_ \otimes ee_3) + y_1 \tau E \circ \chi'' \\
\bar{\chi}'' &= \tau E \circ \chi''.
\end{aligned}$$

Now we give componentwise formulas for $\tilde{x}$ and $\tilde{\tau}$. These formulas are well-defined on $\tilde{E}_{21}$, $\tilde{E}_{22}$, $\tilde{E}_{21}^2$, and $\tilde{E}_{22}^2$ by the lemmas above.

Definition 4.4. We define the action of $\tilde{x}$ on $\tilde{E}$ as follows:

- on $\tilde{E}_{11}$: $\tilde{x}$ acts by x
- on $\tilde{E}_{12}$: $\tilde{x}$ acts by xE
- on $\tilde{E}_{21}$: $\tilde{x}$ acts by $(\theta, \varphi) \mapsto (y\theta, x \circ \varphi)$
- on $\tilde{E}_{22}$: $\tilde{x}$ acts by $(e_1, e_2, \xi) \mapsto (ye_1, xe_2, xE \circ \xi)$.

We define the action of $\tilde{\tau}$ on $\tilde{E}^2$ as follows:

- on $\tilde{E}^2_{11}$: $\tilde{\tau}$ acts by τ
- on $\tilde{E}^2_{12}$: $\tilde{\tau}$ acts by τE
- on $\tilde{E}^2_{21}$: $\tilde{\tau}$ acts by $(e_1, e_2, \xi) \mapsto (e', e', \tau \circ \xi)$
- on $\tilde{E}^2_{22}$: $\tilde{\tau}$ acts by $(ee_1, ee_2, ee_3, \chi) \mapsto (ee', ee', \tau(ee_3), \tau E \circ \chi)$.

Lemma 4.5. *The formulas for $\tilde{x}$ give a (C, C)-bimodule endomorphism of $\tilde{E}$.*

Proof. Recall the definition of the complex E' of (B, B)-bimodules in §3.1.2. There is an $(\begin{pmatrix} A[y] & 0 \\ 0 & A[y] \end{pmatrix}, \begin{pmatrix} A[y] & 0 \\ 0 & A[y] \end{pmatrix})$-bimodule endomorphism x' of E' given componentwise in degrees 0 and 1 by $(A[y], A[y])$-bimodule endomorphisms:

$$x'_0 = \begin{pmatrix} x & xE_y \\ 0 & x \end{pmatrix}, \quad x'_1 = \begin{pmatrix} x & xE_y \\ y & x \end{pmatrix}.$$

The relation $s \circ E_y x = xE_y \circ s$ may be used to check that x'_0 and x'_1 together give a morphism of complexes of (B, B)-bimodules. This map induces a (C, C)-bimodule endomorphism of $\mathrm{Hom}_{K^b(B)}(X, E'X)$ that agrees with the definition of $\tilde{x}$. ☐

It follows that $\tilde{x}$ induces endomorphisms $\tilde{x}\tilde{E}$ and $\tilde{E}\tilde{x}$. For future reference we write the formulas for those:

Proposition 4.6. *The formulas for $\tilde{x}$ determine the following formulas for $\tilde{x}\tilde{E}$ and $\tilde{E}\tilde{x}$ on $\tilde{E}^2$:*

- on $\tilde{E}^2_{11}$: $\tilde{x}\tilde{E}$ acts by xE and $\tilde{E}\tilde{x}$ acts by Ex
- on $\tilde{E}^2_{12}$: $\tilde{x}\tilde{E}$ acts by xE^2 and $\tilde{E}\tilde{x}$ by ExE
- on $\tilde{E}^2_{21}$: $\tilde{x}\tilde{E}$ acts by

$$(e_1, e_2, \xi) \mapsto (ye_1, xe_2, xE \circ \xi)$$

  and $\tilde{E}\tilde{x}$ by

$$(e_1, e_2, \xi) \mapsto (xe_1, ye_2, Ex \circ \xi)$$

- on $\tilde{E}^2_{22}$: $\tilde{x}\tilde{E}$ acts by

$$(ee_1, ee_2, ee_3, \chi) \mapsto (yee_1, xE(ee_2), xE(ee_3), xE^2 \circ \chi)$$

  and $\tilde{E}\tilde{x}$ by

$$(ee_1, ee_2, ee_3, \chi) \mapsto (xE(ee_1), yee_2, Ex(ee_3), ExE \circ \chi).$$

Proof. Use Lemma 3.45, in particular the diagram in the case $n = m = 1$. ☐

4.1.2. *Bimodule structure of $\tilde{E}^2$ and equivariance of $\tilde{\tau}$.*

Lemma 4.7. *The formulas for $\tilde{\tau}$ give a (C,C)-bimodule endomorphism of $\tilde{E}^2$.*

For the maps we defined on components of $\tilde{E}^2$ to determine jointly a (C,C)-bimodule endomorphism $\tilde{\tau}$, they must be equivariant with respect to the left and right C-actions. In order to check equivariance, we write formulas for the actions of the generators in C in the following four lemmas. The reader may verify these formulas from the various definitions.

Lemma 4.8. *Generators in $A[y] \subset C$ act on the right on $\tilde{E}^2$, in terms of the separate bimodule structures of $\tilde{E}^2_{ij}$, as follows:*

- $\tilde{E}^2_{11} \otimes A[y] \to \tilde{E}^2_{11}$ *by*

$$\begin{aligned} y_1y_2E^2[y] \otimes_{A[y]} A[y] &\longrightarrow y_1y_2E^2[y] \\ y_1y_2ee \otimes \theta &\mapsto y_1y_2ee.\theta. \end{aligned}$$

- $\tilde{E}^2_{21} \otimes A[y] \to \tilde{E}^2_{21}$ *by*

$$\begin{aligned} &G_2 \otimes_{A[y]} A[y] \longrightarrow G_2 \\ &(e_1, e_2, \xi) \otimes \theta \mapsto (e_1.\theta, e_2.\theta, \xi(-).\theta)\,. \end{aligned}$$

They act on the left as follows:

- $A[y] \otimes \tilde{E}^2_{11} \to \tilde{E}^2_{11}$ *by*

$$\begin{aligned} A[y] \otimes_{A[y]} y_1y_2E^2[y] &\longrightarrow y_1y_2E^2[y] \\ \theta \otimes y_1y_2ee &\mapsto y_1y_2\theta.ee. \end{aligned}$$

- $A[y] \otimes \tilde{E}^2_{12} \to \tilde{E}^2_{12}$ *by*

$$\begin{aligned} A[y] \otimes_{A[y]} y_1y_2y_3E^3[y] &\longrightarrow y_1y_2y_3E^3[y] \\ \theta \otimes y_1y_2y_3eee &\mapsto y_1y_2y_3\theta.eee. \end{aligned}$$

Remark. We may confirm that the image of the action map $\tilde{E}^2_{21} \to \tilde{E}^2_{21}$ preserves the conditions for G_2:

$$\begin{aligned} \xi.\theta - \_ \otimes e_1.\theta &= y_2\xi_1.\theta, \\ \xi_1.\theta &= \delta(\_ \otimes e_2).\theta + (y_1\xi_2).\theta \\ &= \delta(\_ \otimes e_2.\theta) + y_1(\xi_2.\theta), \end{aligned}$$

and the e_ℓ relation:

$$e_1.\theta - e_2.\theta = y_1e'.\theta.$$

Lemma 4.9. *Generators in $G_1^{\mathsf{op}} \subset C$ act on the right on $\tilde{E}^2$ as follows:*

- $\tilde{E}^2_{12} \otimes G_1^{\mathsf{op}} \to \tilde{E}^2_{12}$ *by*

$$\begin{aligned} y_1y_2y_3E^3[y] \otimes_{G_1^{\mathsf{op}}} G_1^{\mathsf{op}} &\longrightarrow y_1y_2y_3E^3[y] \\ y_1y_2y_3eee \otimes (\theta, \varphi) &\mapsto E^2\varphi(y_1y_2y_3eee) \end{aligned}$$

- $\tilde{E}^2_{22} \otimes G_1^{\mathsf{op}} \to \tilde{E}^2_{22}$ *by*

$$G_3 \otimes_{G_1^{\mathsf{op}}} G_1^{\mathsf{op}} \longrightarrow G_3$$
$$(ee_1, ee_2, ee_3, \chi) \otimes (\theta, \varphi) \mapsto \big(E\varphi(ee_1), E\varphi(ee_2), ee_3.\theta, E^2\varphi \circ \chi\big).$$

They act on the left as follows:

- $G_1^{\mathsf{op}} \otimes \tilde{E}^2_{21} \to \tilde{E}^2_{21}$ *by*

$$G_1^{\mathsf{op}} \otimes_{G_1^{\mathsf{op}}} G_2 \longrightarrow G_2$$
$$(\theta, \varphi) \otimes (e_1, e_2, \xi) \mapsto (\theta.e_1, \theta.e_2, \xi \circ \varphi)$$

- $G_1^{\mathsf{op}} \otimes \tilde{E}^2_{22} \to \tilde{E}^2_{22}$ *by*

$$G_1^{\mathsf{op}} \otimes_{G_1^{\mathsf{op}}} G_3 \longrightarrow G_3$$
$$(\theta, \varphi) \otimes (ee_1, ee_2, ee_3, \chi) \mapsto (\theta.ee_1, \theta.ee_2, \theta.ee_3, \chi \circ \varphi).$$

Remark 4.10. We may confirm that the image of the right action map $\tilde{E}^2_{22} \otimes G_1^{\mathsf{op}} \to \tilde{E}^2_{22}$ preserves the conditions for G_3:

$$\begin{aligned}
E^2\varphi \circ \chi &= _ \otimes E\varphi(ee_1) + E^2\varphi(\chi - _ \otimes ee_1) \\
&= _ \otimes E\varphi(ee_1) + y_3\big(E^2\varphi \circ \chi_1\big), \\
E^2\varphi \circ \chi_1 &= \tau E(_ \otimes E\varphi(ee_2)) + y_2 E^2\varphi \circ \chi_1' \\
&= \tau E \circ E^2(_.\theta + y_1\varphi_1) \circ (_ \otimes ee_2) + y_2 E^2\varphi \circ \chi_1', \\
E^2\varphi \circ \chi_1' &= E^2(_.\theta) \circ E\tau \circ \tau E(_ \otimes ee_3) + y_1 E^2\varphi_1 \circ \chi_1' + y_1\chi''.\theta \\
&= E\tau \circ \tau E(_ \otimes ee_3.\theta) + y_1\big(\chi''.\theta + E^2\varphi_1 \circ \chi_1'\big).
\end{aligned}$$

And the ee_ℓ relations:

$$\begin{aligned}
E\varphi(ee_1) - E\varphi(ee_2) &= y_2 E\varphi(ee'), \\
ee_3.\theta - E\varphi(ee_2) &= (ee_3 - ee_2).\theta - y_1 E\varphi_1(ee_2) \\
&= y_1\left(ee''.\theta - E\varphi_1(ee_2)\right), \\
\delta(ee_3.\theta) - E\varphi(ee_1) &= y_2\tau(ee_3).\theta + (ee_3 - ee_1).\theta - y_1 E\varphi_1(ee_1) \\
&= y_2\tau(ee_3).\theta + y_1 ee''.\theta - y_2 ee'.\theta - y_1 E\varphi_1(ee_1) \\
&= y_1\left(y_2\overline{ee}.\theta + ee''.\theta - E\varphi_1(ee_1)\right).
\end{aligned}$$

Similarly we may confirm that the image of the left action map $G_1^{\mathsf{op}} \otimes \tilde{E}^2_{21} \to \tilde{E}^2_{21}$ lies in G_2:

$$\begin{aligned}
\xi \circ \varphi &= \varphi(-) \otimes e_1 + y_2\xi_1 \circ \varphi \\
&= _ \otimes \theta.e_1 + y_2\big(\varphi_1(-) \otimes e_1 + \xi_1 \circ \varphi\big), \\
\xi_1 \circ \varphi + \varphi_1(-) \otimes e_1 &= \tau(_ \otimes e_2) \circ \varphi + y_1\xi' \circ \varphi + \varphi_1(-) \otimes e_1 \\
&= \tau(_ \otimes \theta.e_2) + \tau y_2(\varphi_1(-) \otimes e_2) + y_1\xi' \circ \varphi + \varphi_1(-) \otimes e_1 \\
&= \tau(_ \otimes \theta.e_2) + y_1\left(\tau(\varphi_1(-) \otimes e_2) + \varphi_1(-) \otimes e' + \xi' \circ \varphi\right).
\end{aligned}$$

And the e_ℓ relation:

$$\theta.e_1 - \theta.e_2 = y_1\theta.e'.$$

And the image of the left action map $G_1^{\mathsf{op}} \otimes \tilde{E}^2_{22} \to \tilde{E}^2_{22}$ lies in G_3:

$$\begin{aligned}
\chi \circ \varphi &= \varphi(-) \otimes ee_1 + y_3 \chi_1 \circ \varphi \\
&= _ \otimes \theta.ee_1 + y_3\big(\varphi_1 \otimes ee_1 + \chi_1 \circ \varphi\big), \\
\chi_1 \circ \varphi &= \tau E(_ \otimes \theta.ee_2) + \tau E y_3(\varphi_1 \otimes ee_2) + y_2 \chi_1' \circ \varphi, \\
\varphi_1 \otimes ee_1 + \chi_1 \circ \varphi &= \tau E(_ \otimes \theta.ee_2) + y_2\Big(\tau E(\varphi_1 \otimes ee_2) + \varphi_1 \otimes ee' + \chi_1' \circ \varphi\Big), \\
\chi_1' \circ \varphi &= E\tau \circ \tau E(_ \otimes \theta.ee_3) + E\tau \circ \tau E \circ y_3(\varphi_1 \otimes ee_3) + y_1 \chi'' \circ \varphi \\
&= E\tau \circ \tau E(_ \otimes \theta.ee_3) + y_1(E\tau \circ \tau E)(\varphi_1 \otimes ee_3) \\
&\quad - \tau E(\varphi_1 \otimes ee_3) - E\tau(\varphi_1 \otimes ee_3) + y_1 \chi'' \circ \varphi,
\end{aligned}$$

$$\begin{aligned}
&\tau E(\varphi_1 \otimes ee_2) + \varphi_1 \otimes ee' + \chi_1' \circ \varphi = \\
&E\tau \circ \tau E(_ \otimes \theta.ee_3) + y_1\Big((E\tau \circ \tau E)(\varphi_1 \otimes ee_3) - \tau E(\varphi_1 \otimes ee'') - \varphi_1 \otimes \overline{ee} + \chi'' \circ \varphi\Big).
\end{aligned}$$

And the ee_ℓ relations:

$$\begin{aligned}
\theta.ee_1 - \theta.ee_2 &= y_2 \theta.ee' \\
\theta.ee_3 - \theta.ee_2 &= y_1 \theta.ee'' \\
\delta(\theta.ee_3) - \theta.ee_1 &= y_1 \theta.ee'''.
\end{aligned}$$

Lemma 4.11. *Generators in $y_1 E[y] \subset C$ act on the right on $\tilde{E}^2$ as follows:*

- $\tilde{E}^2_{11} \otimes y_1 E[y] \to \tilde{E}^2_{12}$ *by*

$$\begin{aligned}
y_1 y_2 E^2[y] \otimes_{A[y]} y_1 E[y] &\longrightarrow y_1 y_2 y_3 E^3[y] \\
y_1 y_2 ee \otimes y_1 e &\mapsto y_1 y_2 y_3 (ee \otimes e)
\end{aligned}$$

- $\tilde{E}^2_{21} \otimes y_1 E[y] \to \tilde{E}^2_{22}$ *by*

$$\begin{aligned}
&G_2 \otimes_{A[y]} y_1 E[y] \longrightarrow G_3 \\
&(e_1, e_2, \xi) \otimes y_1 e \mapsto (e_1 \otimes y_1 e, e_2 \otimes y_1 e, 0, \xi(-) \otimes y_1 e)\,.
\end{aligned}$$

They act on the left as follows:

- $y_1 E[y] \otimes \tilde{E}^2_{21} \to \tilde{E}^2_{11}$ *by*

$$\begin{aligned}
y_1 E[y] \otimes_{G_1^{\mathsf{op}}} G_2 &\longrightarrow y_1 y_2 E^2[y] \\
y_1 e \otimes (e_1, e_2, \xi) &\mapsto \xi(y_1 e)
\end{aligned}$$

- $y_1 E[y] \otimes \tilde{E}^2_{22} \to \tilde{E}^2_{12}$ *by*

$$\begin{aligned}
y_1 E[y] \otimes_{G_1^{\mathsf{op}}} G_3 &\longrightarrow y_1 y_2 y_3 E^3[y] \\
y_1 e \otimes (ee_1, ee_2, ee_3, \chi) &\mapsto \chi(y_1 e).
\end{aligned}$$

Remark. We may confirm that the image of the right action map $\tilde{E}^2_{21} \otimes y_1E[y] \to \tilde{E}^2_{22}$ preserves the conditions for G_3:

$$\begin{aligned} \chi &= \xi \otimes y_1 e, \\ \chi - {}_-\otimes e_1 \otimes y_1 e &= y_1y_3(\xi_1 \otimes e), \\ \chi - \delta E({}_-\otimes e_2 \otimes y_1 e) &= \big(\xi - \delta({}_-\otimes e_2)\big) \otimes y_1 e \\ &= y_1y_2(\xi_2 \otimes e). \end{aligned}$$

Similarly we may confirm that the image of the left action map $y_1E[y]\otimes\tilde{E}^2_{21} \to \tilde{E}^2_{11}$ lies in $y_1y_2E^2[y]$:

$$\begin{aligned} \xi \circ y_1 &= y_2\big({}_-\otimes e_1 + \xi_1 \circ y_1\big), \\ \xi_1 \circ y_1 &= \tau y_2({}_-\otimes e_2) + y_1\xi' \circ y_1 \\ &= y_1\big(\tau({}_-\otimes e_2) + \xi' \circ y_1\big) - {}_-\otimes e_2, \\ \xi \circ y_1 &= y_2\Big(y_1\big(\tau({}_-\otimes e_2) + \xi' \circ y_1\big) + {}_-\otimes (e_1 - e_2)\Big) \\ &= y_1y_2\Big(\tau({}_-\otimes e_2) + {}_-\otimes e' + \xi' \circ y_1\Big). \end{aligned}$$

And the image of the left action map $y_1E[y] \otimes \tilde{E}^2_{22} \to \tilde{E}^2_{12}$ lies in $y_1y_2y_3E^3[y]$:

$$\begin{aligned} \chi \circ y_1 &= y_3\big({}_-\otimes ee_1 + \chi_1 \circ y_1\big) \\ \chi_1 \circ y_1 &= -\tau E y_3 y_1({}_-\otimes ee'') \\ &\quad + E\delta \circ \tau E y_3({}_-\otimes ee_3) + y_1y_2\chi'' \circ y_1 \\ &= -\tau E y_3 y_1({}_-\otimes ee'') + E\delta \circ y_2\tau E({}_-\otimes ee_3) \\ &\quad - E\delta({}_-\otimes ee_3) + y_1y_2\chi'' \circ y_1 \\ &= -y_2\tau E y_1({}_-\otimes ee'') + y_1({}_-\otimes ee'') + y_1y_2E\tau \circ \tau E({}_-\otimes ee_3) \\ &\quad - y_1({}_-\otimes ee''') - {}_-\otimes ee_1 + y_1y_2\chi'' \circ y_1 \\ \chi \circ y_1 &= y_3y_2y_1\Big(-\tau E({}_-\otimes ee'') + E\tau \circ \tau E({}_-\otimes ee_3) + \chi'' \circ y_1\Big) \\ &\quad + y_3y_1\big({}_-\otimes ee'' - {}_-\otimes ee'''\big) \\ &= y_3y_2y_1\Big(-\tau E({}_-\otimes ee'') + E\tau \circ \tau E({}_-\otimes ee_3) - {}_-\otimes \overline{ee} + \chi'' \circ y_1\Big). \end{aligned}$$

Lemma 4.12. *Generators in $F[y] \subset C$ act on the right on $\tilde{E}^2$ as follows:*

- $\tilde{E}^2_{12} \otimes F[y] \to \tilde{E}^2_{11}$ *by*

$$\begin{aligned} y_1y_2y_3E^3[y] \otimes_{G_1^{\mathrm{op}}} F[y] &\longrightarrow y_1y_2E^2[y] \\ y_1y_2y_3eee \otimes f &\mapsto y_1y_2E^2f(y_1eee) \end{aligned}$$

- $\tilde{E}^2_{22} \otimes F[y] \to \tilde{E}^2_{21}$ *by*

$$\begin{aligned} G_3 \otimes_{G_1^{\mathrm{op}}} F[y] &\longrightarrow G_2 \\ (ee_1, ee_2, ee_3, \chi) \otimes f &\mapsto \big(Ef(ee_1), Ef(ee_2), E^2f \circ \chi\big)\,. \end{aligned}$$

They act on the left as follows:

- $F[y] \otimes \tilde{E}^2_{11} \to \tilde{E}^2_{21}$ *by*

$$\begin{aligned} F[y] \otimes_{A[y]} y_1 y_2 E^2[y] &\longrightarrow G_2 \\ f \otimes y_1 y_2 ee &\mapsto (0, 0, f(-).y_1 y_2 ee) \end{aligned}$$

- $F[y] \otimes \tilde{E}^2_{12} \to \tilde{E}^2_{22}$ *by*

$$\begin{aligned} F[y] \otimes_{A[y]} y_1 y_2 y_3 E^3[y] &\longrightarrow G_3 \\ f \otimes y_1 y_2 y_3 eee &\mapsto (0, 0, 0, f(-).y_1 y_2 y_3 eee)\,. \end{aligned}$$

Remark. We may observe that the image of the right action map $\tilde{E}^2_{22} \otimes F[y] \to \tilde{E}^2_{21}$ preserves the conditions for G_3:

$$\begin{aligned} E^2 f \circ \chi - \_ \otimes Ef(ee_1) &= E^2 f \circ (\chi - \_ \otimes ee_1) \\ &= y_2 E^2 f \circ \chi_1, \\ E^2 f \circ \chi - \delta(\_ \otimes Ef(ee_2)) &= E^2 f \circ (\chi - \delta E(\_ \otimes ee_2)) \\ &= E^2 f \circ y_2 \chi_2 \\ &= y_1 E^2 f \circ \chi_2, \end{aligned}$$

and the ee_ℓ relation:

$$\begin{aligned} Ef(ee_1 - ee_2) &= Ef(y_2 ee'') \\ &= y_1 Ef(ee''). \end{aligned}$$

It is trivial to check the conditions for the images of the left action maps $F[y] \otimes \tilde{E}^2_{11} \to \tilde{E}^2_{21}$ and $F[y] \otimes \tilde{E}^2_{12} \to \tilde{E}^2_{22}$.

Proof of Lemma 4.7. The reader may now check that $\tilde{\tau}$ defined in §4.1.1 is equivariant over the left and right C actions. These checks are completely mechanical using the formulas just given. □

4.2. **Hecke relations.**

4.2.1. *$\tilde{x}$ and $\tilde{\tau}$ satisfy Hecke relations.* These checks are also mechanical, but we write them out because they are important.

Proposition 4.13. *On each component $\tilde{E}^2_{ij}$, the maps $\tilde{x}$ and $\tilde{\tau}$ defined in §4.1.1 satisfy*

$$\begin{aligned} \tilde{E}\tilde{x} \circ \tilde{\tau} - \tilde{\tau} \circ \tilde{x}\tilde{E} &= Id \\ \tilde{\tau} \circ \tilde{E}\tilde{x} - \tilde{x}\tilde{E} \circ \tilde{\tau} &= Id. \end{aligned}$$

Proof. On the first row, $\tilde{E}^2_{11}$ and $\tilde{E}^2_{12}$, the relations follow from the corresponding relations between x and τ.

On $\tilde{E}^2_{21}$ presented as G_2, we have:

$$\begin{aligned}
\tilde{E}\tilde{x}\circ\tilde{\tau} &: (e_1,e_2,\xi)\mapsto (xe', ye', Ex\circ\tau\circ\xi)\\
\tilde{\tau}\circ\tilde{x}\tilde{E} &: (e_1,e_2,\xi)\mapsto (ye'-e_2, ye'-e_2, \tau\circ xE\circ\xi)\\
\tilde{\tau}\circ\tilde{E}\tilde{x} &: (e_1,e_2,\xi)\mapsto (e_2+xe', e_2+xe', \tau\circ Ex\circ\xi)\\
\tilde{x}\tilde{E}\circ\tilde{\tau} &: (e_1,e_2,\xi)\mapsto (ye', xe', xE\circ\tau\circ\xi),
\end{aligned}$$

from which

$$\begin{aligned}
\tilde{E}\tilde{x}\circ\tilde{\tau}-\tilde{\tau}\circ\tilde{x}\tilde{E} : (e_1,e_2,\xi)\mapsto \big(y_1e'+e_2, e_2, (Ex\circ\tau-\tau\circ xE)\circ\xi\big)\\
= (e_1,e_2,\xi),
\end{aligned}$$

and similarly for the other relation.

On $\tilde{E}^2_{22}$ presented as G_3, we have:

$$\begin{aligned}
\tilde{E}\tilde{x}\circ\tilde{\tau} &: (ee_1,ee_2,ee_3,\chi)\mapsto (xE(ee'), yee', Ex\circ\tau(ee_3), ExE\circ\tau E\circ\chi)\\
\tilde{\tau}\circ\tilde{x}\tilde{E} &: (ee_1,ee_2,ee_3,\chi)\mapsto (yee'-ee_2, yee'-ee_2, \tau\circ xE(ee_3), \tau E\circ xE^2\circ\chi)\\
\tilde{\tau}\circ\tilde{E}\tilde{x} &: (ee_1,ee_2,ee_3,\chi)\mapsto (ee_1+yee', ee_1+yee', \tau\circ Ex(ee_3), \tau E\circ ExE\circ\chi)\\
\tilde{x}\tilde{E}\circ\tilde{\tau} &: (ee_1,ee_2,ee_3,\chi)\mapsto (yee', xE(ee'), xE\circ\tau(ee_3), xE^2\circ\tau E\circ\chi),
\end{aligned}$$

and so

$$\begin{aligned}
&\tilde{E}\tilde{x}\circ\tilde{\tau}-\tilde{\tau}\circ\tilde{x}\tilde{E} : (ee_1,ee_2,ee_3,\chi)\mapsto\\
&\quad\big(y_2ee'+ee_2, ee_2, (Ex\circ\tau-\tau\circ xE)(ee_3), (ExE\circ\tau E-\tau E\circ xE^2)\circ\chi\big)\\
&\qquad = (ee_1,ee_2,ee_3,\chi),
\end{aligned}$$

and similarly for the other relation. □

4.2.2. $\tilde{\tau}^2=0$. This is clear.

4.2.3. *$\tilde{\tau}$ satisfies the braid relation.* In this section we give formulas defining k-module endomorphisms $\tilde{\tau}_1$ and $\tilde{\tau}_2$ of the components of the matrix parametrization of $\tilde{E}^3$. We show that these endomorphisms satisfy the braid relations. Then we argue that they correspond to the maps $\tilde{E}\tilde{\tau}$ and $\tilde{\tau}\tilde{E}$ induced on the same bimodule components. This will complete our proof that $\tilde{x}$ and $\tilde{\tau}$ satisfy the nil affine Hecke relations in $\mathcal{U}^+$.

Lemma 4.14. *Let us be given $(ee_1,ee_2,ee_3,\chi)\in G_3$ with ee'' defined as in §3.21. Then*

$$(\tau(ee_1), -ee'', -ee'', E\tau\circ\chi)\in E^2[y]^{\oplus 3}\oplus \mathrm{Hom}_A({}_AE, E^3)[y]$$

also lies in G_3.

Proof. The reader may check this directly. In Prop. 4.18 we will interpret this element as the image of (ee_1,ee_2,ee_3,χ) under $\tilde{E}\tilde{\tau}$, and it must therefore lie in G_3. □

Lemma 4.15. *Let us be given* $(eee_1, eee_2, eee_3, eee_4, \psi) \in G_4$ *with* $eee^{(\ell)}$ *defined as in §3.23. Then the following elements of* $E^3[y]^{\oplus 4} \oplus \mathrm{Hom}_A({}_AE, E^4)[y]$ *also lie in* G_4*:*

$$(\tau E(eee_1), eee^{(2)}, eee^{(2)}, E\tau(eee_4), E\tau E \circ \psi),$$
$$(eee^{(4)}, eee^{(4)}, \tau E(eee_3), \tau E(eee_4), \tau E^2 \circ \psi).$$

Proof. The reader may check this directly. In Prop. 4.18 we will interpret these elements as the images of $(eee_1, eee_2, eee_3, eee_4, \psi)$ under $\tilde{E}\tilde{\tau}$ and $\tilde{\tau}\tilde{E}$ respectively, and they must therefore lie in G_4. □

Definition 4.16. Let $\tilde{\tau}_1, \tilde{\tau}_2$ be k-module maps defined on $\tilde{E}^3_{ij}$, presented as in §3.4.2, as follows:

- on $\tilde{E}^3_{11}$:
  - $\tilde{\tau}_1$ acts by $E\tau$
  - $\tilde{\tau}_2$ by τE
- on $\tilde{E}^3_{12}$:
  - $\tilde{\tau}_1$ by $E\tau E$
  - $\tilde{\tau}_2$ by τE^2
- on $\tilde{E}^3_{21}$:
  - $\tilde{\tau}_1$ by $(ee_1, ee_2, ee_3, \chi) \mapsto (\tau(ee_1), -ee'', -ee'', E\tau \circ \chi)$
  - $\tilde{\tau}_2$ by $(ee_1, ee_2, ee_3, \chi) \mapsto (ee', ee', \tau(ee_3), \tau E \circ \chi)$,

    i.e. $\tilde{\tau}$ as defined above on G_3 considered as $\tilde{E}^2_{22}$
- on $\tilde{E}^3_{22}$:
  - $\tilde{\tau}_1$ by $(eee_1, eee_2, eee_3, eee_4, \psi) \mapsto$
    $$(\tau E(eee_1), eee^{(2)}, eee^{(2)}, E\tau(eee_4), E\tau E \circ \psi)$$
  - $\tilde{\tau}_2$ by $(eee_1, eee_2, eee_3, eee_4, \psi) \mapsto$
    $$(eee^{(4)}, eee^{(4)}, \tau E(eee_3), \tau E(eee_4), \tau E^2 \circ \psi).$$

Proposition 4.17. *The* $\tilde{\tau}_i$ *satisfy* $\tilde{\tau}_1 \circ \tilde{\tau}_2 \circ \tilde{\tau}_1 = \tilde{\tau}_2 \circ \tilde{\tau}_1 \circ \tilde{\tau}_2$*.*

Proof. On $\tilde{E}^2_{1j}$ the claim follows from the τ_i braid relation. On $\tilde{E}^2_{21} = G_3$ we have:

$$(ee_1, ee_2, ee_3, \chi) \xmapsto{\tilde{\tau}_1}$$
$$(\tau(ee_1), -ee'', -ee'', E\tau \circ \chi) \xmapsto{\tilde{\tau}_2}$$
$$(-\overline{ee} - \tau(ee'''), -\overline{ee} - \tau(ee'''), -\tau(ee''), \tau E \circ E\tau \circ \chi) \xmapsto{\tilde{\tau}_1}$$
$$(-\tau(\overline{ee}), -\tau(\overline{ee}), -\tau(\overline{ee}), E\tau \circ \tau E \circ E\tau \circ \chi)$$

and

$$(ee_1, ee_2, ee_3, \chi) \xmapsto{\tilde{\tau}_2}$$
$$(ee', ee', \tau(ee_3), \tau E \circ \chi) \xmapsto{\tilde{\tau}_1}$$
$$(\tau(ee'), -\overline{ee}, -\overline{ee}, E\tau \circ \tau E \circ \chi) \xmapsto{\tilde{\tau}_2}$$
$$(-\tau(\overline{ee}), -\tau(\overline{ee}), -\tau(\overline{ee}), \tau E \circ E\tau \circ \tau E \circ \chi).$$

On $\tilde{E}^3_{22} = G_4$ we have:

$$(eee_1, eee_2, eee_3, eee_4, \psi) \xmapsto{\tilde{\tau}_1}$$
$$(\tau E(eee_1), eee^{(2)}, eee^{(2)}, E\tau(eee_4), E\tau E \circ \psi) \xmapsto{\tilde{\tau}_2}$$
$$(\tau E(eee^{(5)}) + \overline{eee}, \tau E(eee^{(5)}) + \overline{eee}, \tau E(eee^{(2)}), \tau E \circ E\tau(eee_4), \tau E^2 \circ E\tau E \circ \psi) \xmapsto{\tilde{\tau}_1}$$
$$(\tau E(\overline{eee}), \tau E(\overline{eee}), \tau E(\overline{eee}), E\tau \circ \tau E \circ E\tau(eee_4), E\tau E \circ \tau E^2 \circ E\tau E \circ \psi)$$

and

$$(eee_1, eee_2, eee_3, eee_4, \psi) \xmapsto{\tilde{\tau}_2}$$
$$(eee^{(4)}, eee^{(4)}, \tau E(eee_3), \tau E(eee_4), \tau E^2 \circ \psi) \xmapsto{\tilde{\tau}_1}$$
$$(\tau E(eee^{(4)}), \overline{eee}, \overline{eee}, E\tau \circ \tau E(eee_4), E\tau E \circ \tau E^2 \circ \psi) \xmapsto{\tilde{\tau}_2}$$
$$(\tau E(\overline{eee}), \tau E(\overline{eee}), \tau E(\overline{eee}), \tau E \circ E\tau \circ \tau E(eee_4), \tau E^2 \circ E\tau E \circ \tau E^2 \circ \psi).$$

□

The remaining goal of this section is to relate the $\tilde{\tau}_i$ just defined to the $\tilde{\tau}$ acting on $\tilde{E}$ as described in §4.1.1. The latter is known to be a (C, C)-bimodule morphism.

Proposition 4.18. *Under the isomorphism of Lemma 3.45, namely*

$$\tilde{E}^3 \xrightarrow{\sim} \mathrm{Hom}_{K^b(B)}(X, E'^3 X),$$

the maps $\tilde{E}\tilde{\tau}$ and $\tilde{\tau}\tilde{E}$ on $\tilde{E}^3$ correspond to $\tilde{\tau}_1$ and $\tilde{\tau}_2$ of Definition 4.16.

Corollary 4.19. *Lemmas 4.14 and 4.15 follow.*

Corollary 4.20. *Proposition 4.17 implies $\tilde{E}\tilde{\tau} \circ \tilde{\tau}\tilde{E} \circ \tilde{E}\tilde{\tau} = \tilde{\tau}\tilde{E} \circ \tilde{E}\tilde{\tau} \circ \tilde{\tau}\tilde{E}$.*

Proof of the proposition. We consider the tensor product $\tilde{E} \otimes_C \tilde{E}^2$ formed according to the procedure of §2.4, and study the endofunctor $\tilde{E}\tilde{\tau}$ as in Lemma 2.8, and similarly for $\tilde{E}^2 \otimes_C \tilde{E}$ and $\tilde{\tau}\tilde{E}$. From Lemma 3.45, we have isomorphisms

$$\mathrm{Hom}_{K^b(B)}(X, E'X) \otimes_C \mathrm{Hom}_{K^b(B)}(X, E'^2 X) \xrightarrow{\sim} \mathrm{Hom}_{K^b(B)}(X, E'^3 X)$$
$$\mathrm{Hom}_{K^b(B)}(X, E'^2 X) \otimes_C \mathrm{Hom}_{K^b(B)}(X, E'X) \xrightarrow{\sim} \mathrm{Hom}_{K^b(B)}(X, E'^3 X)$$

associated with the products

$$\tilde{E} \otimes_C \tilde{E}^2 = \tilde{E}^3$$
$$\tilde{E}^2 \otimes_C \tilde{E} = \tilde{E}^3.$$

The maps are given by

$$f \otimes g \mapsto E'g \circ f$$
$$f \otimes g \mapsto E'^2 g \circ f.$$

These isomorphisms determine actions of $\tilde{E}\tilde{\tau}$ and $\tilde{\tau}\tilde{E}$ on $\mathrm{Hom}_{K^b(B)}(X, E'^3 X)$ that we may compare to the $\tilde{\tau}_1$ and $\tilde{\tau}_2$ defined there by components.

The components $\tilde{E}_{ij}$ and $\tilde{E}^2_{ij}$ are $(\mathrm{End}(X_i)^{\mathrm{op}}, \mathrm{End}(X_j)^{\mathrm{op}})$-bimodules, and $\tilde{\tau}$ gives bimodule endomorphisms $\tilde{\tau}_{|ij}$ of the latter. These induce endomorphisms $(\tilde{E}\tilde{\tau})^{1|2}_{|ijk}$ of

$$\tilde{E}^{1|2}_{ijk} = \tilde{E}_{ij} \otimes_{\mathrm{End}(X_j)^{\mathrm{op}}} \tilde{E}^2_{jk},$$

as in §2.4. We know that $\tilde{E}^3_{ik}$ is canonically isomorphic to a quotient of $\tilde{E}^{1|2}_{i1k} \oplus \tilde{E}^{1|2}_{i2k}$, and that $\begin{pmatrix} (\tilde{E}\tilde{\tau})^{1|2}_{|i1k} & 0 \\ 0 & (\tilde{E}\tilde{\tau})^{1|2}_{|i2k} \end{pmatrix}$ acting on $\tilde{E}^{1|2}_{i1k} \oplus \tilde{E}^{1|2}_{i2k}$ descends to $\tilde{E}^3_{ik}$, where it gives the components of $\tilde{E}\tilde{\tau}$. Here it may be compared directly with $\tilde{\tau}_1$ that we defined on $\tilde{E}^3_{ik}$. It therefore suffices for our objective to check commutativity of the following diagrams labeled $D_{1|2}(i,j,k)$, indexed by triples $(i,j,k) \in \{1,2\}^3$:

$$D_{1|2}(i,j,k): \quad \begin{array}{ccc} \tilde{E}_{ij} \otimes_{\mathrm{End}(X_j)^{\mathrm{op}}} \tilde{E}^2_{jk} & \xrightarrow[f\otimes g \mapsto E'g\circ f]{} & \tilde{E}^3_{ik} \\ \Big\downarrow (\tilde{E}\tilde{\tau})^{1|2}_{|ijk} & & \Big\downarrow \tilde{\tau}_{1|ik} \\ \tilde{E}_{ij} \otimes_{\mathrm{End}(X_j)^{\mathrm{op}}} \tilde{E}^2_{jk} & \xrightarrow{f\otimes g \mapsto E'g\circ f} & \tilde{E}^3_{ik}. \end{array}$$

Exactly parallel considerations apply to the study of $\tilde{\tau}\tilde{E}$, where the diagrams for (i,j,k), now labeled $D_{2|1}(i,j,k)$, instead involve maps $(\tilde{E}\tilde{\tau})^{2|1}_{|ijk}$ and $\tilde{\tau}_{2|ik}$.

Checking the diagrams will occupy the next three pages.

Lemma 4.21. *The diagrams $D_{1|2}(i,j,k)$ commute.*

Proof. We consider the diagrams in turn:

- Diagram $D_{1|2}(1,1,1)$:

Consider $(\tilde{E}\tilde{\tau})^{1|2}_{|111} \in \mathrm{End}(\tilde{E}_{11} \otimes \tilde{E}^2_{11})$. Let $y_1 e \in \tilde{E}_{11}$ and $y_1y_2ee \in \tilde{E}^2_{11}$. The image of $y_1e \otimes y_1y_2ee$ in the top right corner of the diagram is

$$E'(y_1y_2ee) \circ y_1 e = y_1y_2y_3(e \otimes ee) \in \tilde{E}^3_{11}.$$

Here we can write out $E'(y_1y_2ee) = (y_1y_2ee, 0, 0, \_ \otimes y_1y_2ee) \in G_3$. On the other hand, $\tilde{\tau}(y_1y_2ee) = y_1y_2\tau(ee)$, so the image of $(\tilde{E}\tilde{\tau})^{1|2}_{|111}\big(y_1e \otimes y_1y_2ee\big)$ is $y_1y_2y_3(e \otimes \tau(ee)) \in \tilde{E}^3_{11}$, which agrees with $\tilde{\tau}_1(y_1y_2y_3(e \otimes ee))$.

- Diagram $D_{1|2}(1,2,1)$:

Consider $(\tilde{E}\tilde{\tau})^{1|2}_{|121} \in \mathrm{End}(\tilde{E}_{12} \otimes \tilde{E}^2_{21})$. Let $y_1y_2ee \in \tilde{E}_{12}$ and $(e_1, e_2, \xi) \in \tilde{E}^2_{21}$. We have no established notation for $E'\big((e_1,e_2,\xi)\big) \in \mathrm{Hom}_{K^b(B)}(E'X_2, E'^2X_1)$. It is nevertheless easy to check, by tracking 'leading terms' of the upper rows, that

$$E'\big((e_1,e_2,\xi)\big) \circ y_1y_2ee = E\xi(y_1y_2ee) \in \tilde{E}^3_{11}.$$

This lies in $y_1y_2y_3E^3[y]$. Then $\tilde{\tau}\big((e_1,e_2,\xi)\big) = (e', e', \tau \circ \xi)$, so $(\tilde{E}\tilde{\tau})^{1|2}_{|121}$ applied to $y_1y_2ee \otimes (e_1,e_2,\xi)$ and viewed in $\tilde{E}^3_{11}$ is $E\tau \circ E\xi(y_1y_2ee)$.

- Diagram $D_{1|2}(2,1,1)$:

Consider $(\tilde{E}\tilde{\tau})^{1|2}_{|211} \in \mathrm{End}(\tilde{E}_{21} \otimes \tilde{E}^2_{11})$. Let $(\theta, \varphi) \in \tilde{E}_{21}$ and $y_1y_2ee \in \tilde{E}^2_{11}$. This time we can write $E'(y_1y_2ee) = (y_1y_2ee, 0, 0, \_ \otimes y_1y_2ee)$. Then

$$E'(y_1y_2ee) \circ (\theta, \varphi) = (\theta y_1y_2ee, 0, 0, \varphi \otimes y_1y_2ee) \in \tilde{E}^3_{21}.$$

Going around the diagram in either direction yields $(\theta y_1y_2\tau(ee), 0, 0, \varphi \otimes y_1y_2\tau(ee))$.

- Diagram $D_{1|2}(2, 2, 1)$:

Consider $(\tilde{E}\tilde{\tau})^{1|2}_{|221} \in \mathrm{End}(\tilde{E}_{22} \otimes \tilde{E}^2_{21})$. Let $(e_1, e_2, \xi) \in \tilde{E}_{22}$ and $(\bar{e}_1, \bar{e}_2, \bar{\xi}) \in \tilde{E}^2_{21}$. We have no notation for $E'\left((\bar{e}_1, \bar{e}_2, \bar{\xi})\right)$. One computes that

$$E'\left((\bar{e}_1, \bar{e}_2, \bar{\xi})\right) \circ (e_1, e_2, \xi) = (\bar{\xi}(e_1), e_2 \otimes \bar{e}_1, e_2 \otimes \bar{e}_2, E\bar{\xi} \circ \xi) \in \tilde{E}^3_{21}.$$

Traversing the diagram in either direction gives $(\tau \circ \bar{\xi}(e_1), e_2 \otimes \bar{e}', e_2 \otimes \bar{e}', E\tau \circ E\bar{\xi} \circ \xi)$.

- Diagram $D_{1|2}(1, 1, 2)$:

Consider $(\tilde{E}\tilde{\tau})^{1|2}_{|112} \in \mathrm{End}(\tilde{E}_{11} \otimes \tilde{E}^2_{12})$. Let $y_1e \in \tilde{E}_{11}$ and $y_1y_2y_3eee \in \tilde{E}^2_{12}$. Again by tracking 'leading terms', one checks that

$$E'(y_1y_2y_3eee) \circ y_1e = y_1 \ldots y_4(e \otimes eee) \in \tilde{E}^3_{12}.$$

Traversing the diagram in either direction gives $E\tau E(y_1 \ldots y_4 e \otimes eee)$ which is $y_1 \ldots y_4(e \otimes \tau E(eee))$.

- Diagram $D_{1|2}(1, 2, 2)$:

Consider $(\tilde{E}\tilde{\tau})^{1|2}_{|122} \in \mathrm{End}(\tilde{E}_{12} \otimes \tilde{E}^2_{22})$. Let $y_1y_2ee \in \tilde{E}_{12}$ and $(ee_1, ee_2, ee_3, \chi) \in \tilde{E}^2_{22}$. Then check that

$$E'\left((ee_1, ee_2, ee_3, \chi)\right) \circ y_1y_2ee = E\chi(y_1y_2ee) \in \tilde{E}^3_{12}.$$

Traversing the diagram in either direction gives $(E\tau E \circ E\chi)(y_1y_2ee)$.

- Diagram $D_{1|2}(2, 1, 2)$:

Consider $(\tilde{E}\tilde{\tau})^{1|2}_{|212} \in \mathrm{End}(\tilde{E}_{21} \otimes \tilde{E}^2_{12})$. Let $(\theta, \varphi) \in \tilde{E}_{21}$ and $y_1y_2y_3eee \in \tilde{E}^2_{12}$. Then check that

$$E'(y_1y_2y_3eee) \circ (\theta, \varphi) = (\theta y_1y_2y_3eee, 0, 0, 0, \varphi \otimes y_1y_2y_3eee) \in \tilde{E}^3_{22}.$$

Traversing the diagram in either direction gives

$$(\tau E(\theta y_1y_2y_3eee), 0, 0, 0, E\tau E \circ (\varphi \otimes y_1y_2y_3eee)).$$

- Diagram $D_{1|2}(2, 2, 2)$:

Consider $(\tilde{E}\tilde{\tau})^{1|2}_{|222} \in \mathrm{End}(\tilde{E}_{22} \otimes \tilde{E}^2_{22})$. Let $(e_1, e_2, \xi) \in \tilde{E}_{22}$ and $(ee_1, ee_2, ee_3, \chi) \in \tilde{E}^2_{22}$. Then check that

$$E'\left((ee_1, ee_2, ee_3, \chi)\right) \circ (e_1, e_2, \xi) = (\chi(e_1), e_2 \otimes ee_1, e_2 \otimes ee_2, e_2 \otimes ee_3, E\chi \circ \xi) \in \tilde{E}^3_{22}.$$

Traversing the diagram in either direction gives

$$\left(\tau E(\chi(e_1)), e_2 \otimes ee', e_2 \otimes ee', E\tau(e_2 \otimes ee_3), E\tau E \circ E\chi \circ \xi\right).$$

□

Lemma 4.22. *The diagrams $D_{2|1}(i, j, k)$ commute.*

Proof. We consider the diagrams in turn:

- Diagram $D_{2|1}(1,1,1)$:

Consider $(\tilde{\tau}\tilde{E})^{2|1}_{|111} \in \mathrm{End}(\tilde{E}^2_{11} \otimes \tilde{E}_{11})$. Let $y_1y_2ee \in \tilde{E}^2_{11}$ and $y_1e \in \tilde{E}_{11}$. Then check that

$$E'^2(y_1e) \circ y_1y_2ee = y_1y_2y_3ee \otimes e \in \tilde{E}^3_{11}.$$

Traversing the diagram in either direction gives

$$y_1y_2y_3(\tau(ee) \otimes e).$$

- Diagram $D_{2|1}(1,2,1)$:

Consider $(\tilde{\tau}\tilde{E})^{2|1}_{|121} \in \mathrm{End}(\tilde{E}^2_{12} \otimes \tilde{E}_{21})$. Let $y_1y_2y_3eee \in \tilde{E}^2_{12}$ and $(\theta,\varphi) \in \tilde{E}_{21}$. Then check that

$$E'^2\big((\theta,\varphi)\big) \circ y_1y_2y_3eee = E^2\varphi(y_1y_2y_3eee) \in \tilde{E}^3_{11}.$$

Traversing the diagram in either direction gives

$$(\tau E \circ E^2\varphi)(y_1y_2y_3eee).$$

- Diagram $D_{2|1}(2,1,1)$:

Consider $(\tilde{\tau}\tilde{E})^{2|1}_{|211} \in \mathrm{End}(\tilde{E}^2_{21} \otimes \tilde{E}_{11})$. Let $(e_1,e_2,\xi) \in \tilde{E}^2_{21}$ and $y_1e \in \tilde{E}_{11}$. Then check that

$$E'^2(y_1e) \circ (e_1,e_2,\xi) = (e_1 \otimes y_1e, e_2 \otimes y_1e, 0, \xi \otimes y_1e) \in \tilde{E}^3_{21}.$$

Traversing the diagram in either direction gives

$$(e' \otimes y_1e, e' \otimes y_1e, 0, (\tau \circ \xi) \otimes y_1e).$$

- Diagram $D_{2|1}(2,2,1)$:

Consider $(\tilde{\tau}\tilde{E})^{2|1}_{|221} \in \mathrm{End}(\tilde{E}^2_{22} \otimes \tilde{E}_{21})$. Let $(ee_1,ee_2,ee_3,\chi) \in \tilde{E}^2_{22}$ and $(\theta,\varphi) \in \tilde{E}_{21}$. Then check that

$$E'^2\big((\theta,\varphi)\big) \circ (ee_1,ee_2,ee_3,\chi) = (E\varphi(ee_1), E\varphi(ee_2), \theta ee_3, E^2\varphi \circ \chi) \in \tilde{E}^3_{21}.$$

Traversing the diagram in either direction gives

$$\big(E\varphi(ee'), E\varphi(ee'), \theta\tau(ee_3), E^2\varphi \circ \tau E \circ \chi\big).$$

- Diagram $D_{2|1}(1,1,2)$:

Consider $(\tilde{\tau}\tilde{E})^{2|1}_{|112} \in \mathrm{End}(\tilde{E}^2_{11} \otimes \tilde{E}_{12})$. Let $y_1y_2ee \in \tilde{E}^2_{11}$ and $y_1y_2\bar{e}\bar{e} \in \tilde{E}_{12}$. Then check that

$$E'^2(y_1y_2\bar{e}\bar{e}) \circ y_1y_2ee = (y_1y_2ee) \otimes (y_1y_2\bar{e}\bar{e}) = y_1 \dots y_4(ee \otimes \bar{e}\bar{e}) \in \tilde{E}^3_{12}.$$

Traversing the diagram in either direction gives

$$y_1 \dots y_4(\tau(ee) \otimes \bar{e}\bar{e}).$$

- Diagram $D_{2|1}(1,2,2)$:

Consider $(\tilde\tau\tilde E)^{2|1}_{|122} \in \mathrm{End}(\tilde E^2_{12} \otimes \tilde E_{22})$. Let $y_1y_2y_3eee \in \tilde E^2_{12}$ and $(e_1, e_2, \xi) \in \tilde E_{22}$. Then check that

$$E'^2\big((e_1, e_2, \xi)\big) \circ y_1y_2y_3eee = E^2\xi(y_1y_2y_3eee) \in \tilde E^3_{12}.$$

Traversing the diagram in either direction gives

$$(\tau E^2 \circ E^2\xi)(y_1y_2y_3eee).$$

- Diagram $D_{2|1}(2,1,2)$:

Consider $(\tilde\tau\tilde E)^{2|1}_{|212} \in \mathrm{End}(\tilde E^2_{21} \otimes \tilde E_{12})$. Let $(e_1, e_2, \xi) \in \tilde E^2_{21}$ and $y_1y_2ee \in \tilde E_{12}$. Then check that

$$E'^2(y_1y_2ee) \circ (e_1, e_2, \xi) = (e_1 \otimes y_1y_2ee, e_2 \otimes y_1y_2ee, 0, 0, \xi \otimes y_1y_2ee) \in \tilde E^3_{22}.$$

Traversing the diagram in either direction gives

$$(e' \otimes y_1y_2ee, e' \otimes y_1y_2ee, 0, 0, (\tau \circ \xi) \otimes y_1y_2ee)\,.$$

- Diagram $D_{2|1}(2,2,2)$:

Consider $(\tilde\tau\tilde E)^{2|1}_{|222} \in \mathrm{End}(\tilde E^2_{22}\otimes\tilde E_{22})$. Let $(ee_1, ee_2, ee_3, \chi) \in \tilde E^2_{22}$ and $(e_1, e_2, \xi) \in \tilde E_{22}$. Then check that

$$E'^2\big((e_1, e_2, \xi)\big) \circ (ee_1, ee_2, ee_3, \chi) = \\ \big(E\xi(ee_1), E\xi(ee_2), ee_3 \otimes e_1, ee_3 \otimes e_2, E^2\xi \circ \chi\big) \in \tilde E^3_{22}.$$

Traversing the diagram in either direction gives

$$\big(E\xi(ee'), E\xi(ee'), \tau(ee_3) \otimes e_1, \tau(ee_3) \otimes e_2, \tau E^2 \circ E^2\xi \circ \chi\big)\,.$$

□

The proposition that $\tilde E\tilde\tau$ and $\tilde\tau\tilde E$ correspond to $\tilde\tau_1$ and $\tilde\tau_2$ is now proved. □

4.3. Definition of $\mathcal{L}(1)\bigotimes\mathcal{V}$.

Definition 4.23. Let $\mathcal{V}$ be a 2-representation of $\mathcal{U}^+$ given by the data (E, x, τ) for a k-algebra A such that ${}_AE$ is finitely generated and projective and E^n is free as a P_n-module. We define $\mathcal{L}(1) \bigotimes \mathcal{V}$ to be the 2-representation of $\mathcal{U}^+$ given for the k-algebra C by the data $(\tilde E, \tilde x, \tilde\tau)$.

Proposition 4.24. *If E is locally nilpotent, then $\tilde E$ is locally nilpotent.*

Proof. Note that in our setting of bimodules, local nilpotence of $E \otimes_A -$ is equivalent to nilpotence of E, meaning that $E^n \cong 0$ for some n. This is because local nilpotence implies $E^n \otimes_A A \cong 0$ for some n, but that is just E^n as a bimodule.

Recall the expression for $\tilde E^n$ as a matrix of $(A[y], A[y])$-bimodules:

$$\begin{pmatrix} y_1 \dots y_n E^n[y] & y_1 \dots y_{n+1} E^{n+1}[y] \\ G_n & G_{n+1} \end{pmatrix} \xrightarrow{\sim} \begin{pmatrix} \tilde E^n_{11} & \tilde E^n_{12} \\ \tilde E^n_{21} & \tilde E^n_{22} \end{pmatrix}.$$

The method we used to compute a model for G_n for $n = 1, 2, 3$ also shows that G_n for any n can be described as a sub-bimodule of $E^{n-1}[y]^{\oplus n} \oplus \mathrm{Hom}_A({}_AE, E^n)[y]$, given by the elements satisfying a certain set of conditions. It follows that G_n

vanishes for large n if E^n does. Also $y_1 \dots y_n E^n[y]$ vanishes for large n because E^n does. It follows that $\tilde{E}$ is nilpotent. □

4.3.1. *Weights and gradings for $\mathcal{L}(1) \bigotimes \mathcal{V}$.* It frequently happens that a 2-representation has additional structure, and we may ask whether our 2-product inherits that structure. A 2-representation of $\mathcal{U}^+$ may have a weight decomposition, or its algebra may have a grading.

Definition 4.25. A 2-representation $\mathcal{V}$ of $\mathcal{U}^+$ given for k-algebra A by the data (E, x, τ) is said to have a weight decomposition when A has the form $A = \prod_{i \in \mathbb{Z}} A_i$ with units $e_i \in A_i$, and $e_j E e_i = \delta_{i+2,j} \cdot e_{i+2} E e_i$.

Proposition 4.26 (weight decomposition)**.** *Let A and (E, x, τ) satisfy the conditions of Def. 4.23, and let $\mathcal{V}$ be the 2-representation they determine. Suppose that $\mathcal{V}$ has a weight decomposition with units $e_i \in A_i$. Let C and $(\tilde{E}, \tilde{x}, \tilde{\tau})$ give the data of $\mathcal{L}(1) \bigotimes \mathcal{V}$. Then C has a weight decomposition $C = \prod_{i \in \mathbb{Z}} C_i$ with $C_i = f_i C f_i$ where the units $f_i \in C_i \subset C$ are given in matrix form as follows:*

$$f_i = \begin{pmatrix} e_{i+1} & 0 \\ 0 & (e_{i-1}, \_.e_{i-1}) \end{pmatrix}.$$

Proof. The elements f_i are clearly idempotent and orthogonal, and they sum to the identity. We have for the matrix components of $f_j \tilde{E} f_i$:

$$[f_j \tilde{E} f_i]_{11} = e_{j+1}.y_1 E[y].e_{i+1}$$

$$[f_j \tilde{E} f_i]_{12} = e_{j+1}.y_1 y_2 E^2[y].e_{i-1}$$

$$[f_j \tilde{E} f_i]_{21} = G_1 \bigcap \Big(e_{j-1} A[y] e_{i+1} \oplus e_{j-1}.\mathrm{Hom}_A({}_A E, E).e_{i+1}[y] \Big)$$

$$[f_j \tilde{E} f_i]_{22} = G_2 \bigcap \Big(e_{j-1}.E[y].e_{i-1} \oplus e_{j-1}.E[y].e_{i-1} \oplus e_{j-1}.\mathrm{Hom}_A({}_A E, E^2).e_{i-1}[y] \Big).$$

These are all zero unless $j = i + 2$. □

Definition 4.27 (graded case)**.** A 2-representation $\mathcal{V}$ of $\mathcal{U}^+$ given for k-algebra A by the data (E, x, τ) is said to be a $\mathbb{Z}$-graded 2-representation when A is a $\mathbb{Z}$-graded k-algebra, E is a graded bimodule, and x and τ are graded endomorphisms with $\deg x = +2$ and $\deg \tau = -2$.

Proposition 4.28. *Let A and (E, x, τ) satisfy the conditions of Def. 4.23, and let $\mathcal{V}$ be the 2-representation they determine. Suppose that $\mathcal{V}$ is a $\mathbb{Z}$-graded 2-representation. Let C and $(\tilde{E}, \tilde{x}, \tilde{\tau})$ give the data of $\mathcal{L}(1) \bigotimes \mathcal{V}$. Then $\mathcal{L}(1) \bigotimes \mathcal{V}$ is a $\mathbb{Z}$-graded 2-representation. The gradings on generators in C and $\tilde{E}$ are inherited from the gradings on A and E with the assumption that $\deg y = +2$.*

Proof. It is trivial to check that C is graded and $\tilde{E}$ is a graded bimodule. The formulas for $\tilde{x}$ and $\tilde{\tau}$ in Def. 4.4 show that they have the right degrees. □

5. Comparison: $\mathcal{V} = \mathcal{L}(1)$

In §5.1 we describe a well-known 2-representation of $\mathcal{U}^+$ categorifying $L(1) \otimes L(1)$ using Soergel bimodules. In §5.2 we describe our product explicitly for $\mathcal{V} = \mathcal{L}(1)$, and in §5.3 we show that the result is equivalent to the known one. The reader is warned that notations in this section will diverge from the previous sections.

Let $P_2 = k[y_1, y_2]$. Let S_2 denote the symmetric group on 2 letters, generated by t_1, and acting on P_2 by permutation of the y_i. Let $P_2^{S_2}$ be the subalgebra generated by invariant homogeneous polynomials.

5.1. A categorification of $L(1) \otimes L(1)$.

Definition 5.1. We define:

- a (P_2, P_2)-bimodule $B_{s_1} = P_2 \otimes_{P_2^{S_2}} P_2$
  - and observe that B_{s_1} is also a P_2-algebra with structure map $P_2 \to B_{s_1}$ given by $f \mapsto 1 \otimes f$
  - and that P_2 is a left B_{s_1}-module by $(f \otimes g).\theta = fg\theta$
- a P_2-algebra $T = T_{+2} \oplus T_0 \oplus T_{-2}$ by

$$T_{+2} = P_2,\ T_0 = \mathrm{End}_{B_{s_1}}(P_2 \oplus B_{s_1})^{\mathsf{op}},\ T_{-2} = P_2$$

- a (T, T)-bimodule $\mathscr{E} =_{+2} \mathscr{E}_0 \oplus {}_0\mathscr{E}_{-2}$ by

$${}_0\mathscr{E}_{-2} = \begin{pmatrix} P_2 \\ B_{s_1} \end{pmatrix} \cong T_0 e_2$$

$${}_{+2}\mathscr{E}_0 = \begin{pmatrix} P_2 & B_{s_1} \end{pmatrix} \cong e_2 T_0$$

  for e_2 the projection onto B_{s_1}
  - and observe the canonical isomorphism

$${}_{+2}\mathscr{E}^2_{-2} = e_2 T_0 \otimes_{T_0} T_0 e_2 \xrightarrow{\sim} B_{s_1}$$

- a bimodule endomorphism $x \in \mathrm{End}(\mathscr{E})$ by

$${}_{+2}x_0 = \begin{pmatrix} y_2 & y_2 \otimes 1 \end{pmatrix}, \quad {}_0x_{-2} = \begin{pmatrix} y_1 \\ y_1 \otimes 1 \end{pmatrix}$$

  (acting by multiplication)
- a bimodule endomorphism $\tau \in \mathrm{End}(\mathscr{E}^2)$ by

$${}_{+2}\tau_{-2} : f \otimes g \mapsto \partial_{t_1}(f) \otimes g$$

  where $\partial_{t_1} \in \mathrm{End}_k(P_2)$ is a Demazure operator:

$$\partial_{s_1} : f \mapsto \frac{f - f^{t_1}}{y_1 - y_2}.$$

The next theorem is well-known. Cf., for example, Lauda [Lau09], Webster [Web16, §2.3], Stroppel [Str03, §5.1.1], Sartori-Stroppel [SS15]:

Theorem 5.2. *The k-algebra T and triple $(\mathscr{E}, x, \tau)$ defined above gives a 2-representation of $\mathcal{U}^+$, called $\mathscr{T}$ below, that categorifies the tensor product $L(1) \otimes L(1)$ of fundamental representations of $\mathfrak{sl}_2$.*

5.2. $\mathcal{L}(1)\circledast\mathcal{L}(1)$. We notate both factors as in §2.2.3 except that on the right factor we use y_1 in place of y, and on the left factor we use y_2 in place of y. We write E_i, x_i, τ_i, $i = 1, 2$ for the 2-representation data on the right ($i = 1$) and on the left ($i = 2$).

In the formulas we have given for the product, the algebra A, now A_1, becomes $k[y_1]_{+1} \times k[y_1]_{-1}$ (in its weight decomposition), E becomes $k[y_1]$, x becomes y_1, and y becomes y_2. Let $\omega = y_1 - y_2 \in P_2$. So ω will take over the role of '$y_1 = x-y$' that was written in previous sections. Write $\pi : P_2 \to P_2/(\omega)$ for the projection.

We let B, X, E', C, $\tilde{E}$, $\tilde{x}$, and $\tilde{\tau}$ be defined as above. The algebra B and complex X have nonzero elements only in weights -2, 0, $+2$. These are given as follows:

$$B_{-2} = \begin{pmatrix} P_2 & 0 \\ 0 & 0 \end{pmatrix}, \qquad X_{1_{-2}} = \begin{pmatrix} P_2 \\ 0 \end{pmatrix}, \qquad X_{2_{-2}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$
$$B_0 = \begin{pmatrix} P_2 & k[y] \\ 0 & P_2 \end{pmatrix}, \qquad X_{1_0} = \begin{pmatrix} P_2 \\ 0 \end{pmatrix}, \qquad X_{2_0} = \begin{pmatrix} P_2 \xrightarrow{\pi} P_2/(\omega) \\ 0 \to P_2 \end{pmatrix},$$
$$B_{+2} = \begin{pmatrix} 0 & 0 \\ 0 & P_2 \end{pmatrix}, \qquad X_{1_{+2}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad X_{2_{+2}} = \begin{pmatrix} 0 \\ 0 \to P_2 \end{pmatrix}.$$

Here the action of $P_2/(\omega)$ from the upper right of B_0 on X_{2_0} is $P_2/(\omega) \otimes_{P_2} P_2 \to P_2/(\omega)$ given by $f \otimes 1 \mapsto f$. The complexes for X start in degree 0 on the left. The matrix coefficients are in each case from the -1 weight space of A_2 in the upper left corner.

To compute $\tilde{E}$ we will also need $E'X_2$, which is:

$${}_0E'_{-2}(X_{2_{-2}}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$
$${}_{+2}E'_0(X_{2_0}) = \begin{pmatrix} 0 \\ 0 \to P_2 \oplus P_2 \xrightarrow{(-\pi,\pi)} P_2/(\omega) \end{pmatrix}.$$

Next we compute C:

$$[C_{+2}] = \begin{pmatrix} 0 & 0 \\ 0 & P_2 \end{pmatrix}, \; [C_0] = \begin{pmatrix} P_2 & \omega P_2 \\ P_2 & Q_1^{\mathsf{op}} \end{pmatrix}, \; [C_{-2}] = \begin{pmatrix} P_2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Here $Q_1^{\mathsf{op}} \subset P_2 \oplus P_2$ is the (commutative) algebra of all (θ, φ) such that $\varphi - \theta \in \omega P_2$, with componentwise multiplication. It is a P_2-algebra by $P_2 \ni f \mapsto (f, f) \in Q_1$. The algebra structure of C_0 (cf. §2.4) may be described as follows. The upper right term, ωP_2, is a left P_2-module by multiplication. It is a right Q_1^{op}-module with (θ, φ) acting by multiplication by φ. The lower left P_2 is a left Q_1^{op}-module with the same action. It has a right P_2 action by multiplication. The remaining structure maps are:

$$\omega P_2 \otimes_{P_2} P_2 \to P_2 \quad \text{by } \omega\theta' \otimes \theta \mapsto \omega\theta\theta' \tag{5.1}$$

and

$$P_2 \otimes_{P_2} \omega P_2 \to Q_1^{\mathsf{op}} \quad \text{by } \theta \otimes \omega\theta' \mapsto (0, \omega\theta\theta'). \tag{5.2}$$

Now compute $\tilde{E}$ and the endomorphisms $\tilde{x}$ by components:

$$_0[\tilde{E}]_{-2} = \begin{pmatrix} \omega P_2 & 0 \\ Q_1 & 0 \end{pmatrix}, \qquad _0[\tilde{x}]_{-2} = \begin{pmatrix} y_1 & 0 \\ (y_2, y_1) & 0 \end{pmatrix},$$

$$_{+2}[\tilde{E}]_0 = \begin{pmatrix} 0 & 0 \\ P_2 & Q_2 \end{pmatrix}, \qquad _{+2}[\tilde{x}]_0 = \begin{pmatrix} 0 & 0 \\ y_2 & (y_2, y_1) \end{pmatrix},$$

where $Q_2 \subset P_2 \oplus P_2$ is the (P_2, Q_1^{op})-bimodule containing all (e_1, e_2) such that $e_1 - e_2 \in \omega P_2$; Q_1^{op} acts on Q_2 on the right by $(e_1, e_2).(\theta, \varphi) = (e_1\varphi, e_2\theta)$ (note the swap), and P_2 on the left by diagonal multiplication.

In the next section it will be useful to view $_0\tilde{E}_{-2}$ as $C_0 q_2$ using the idempotent $q_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in [C_0]$, and to view $_{+2}\tilde{E}_0$ as $q_2 C_0$ using the isomorphism of (P_2, Q_1^{op})-bimodules $\sigma : Q_1 \xrightarrow{\sim} Q_2$ by $(\theta, \varphi) \mapsto (\varphi, \theta)$. Viewing them in this way, we may write $_0\tilde{x}_{-2}$ as multiplication on $C_0 q_2$ on the left by $\left(\begin{smallmatrix} y_1 & 0 \\ 0 & (y_2, y_1) \end{smallmatrix}\right) \in C_0$, and $_{+2}\tilde{x}_0$ as multiplication on $q_2 C_0$ on the right by $\left(\begin{smallmatrix} y_2 & 0 \\ 0 & (y_1, y_2) \end{smallmatrix}\right) \in C_0$ (note the swap).

Finally, compute $\tilde{E}^2$ and $\tilde{\tau}$ by components:

$$_{+2}[\tilde{E}^2]_{-2} = \begin{pmatrix} 0 & 0 \\ Q_2 & 0 \end{pmatrix}, \qquad _{+2}[\tilde{\tau}]_{-2} = \begin{pmatrix} 0 & 0 \\ t_{21} & 0 \end{pmatrix},$$

where

$$t_{21} : (e_1, e_2) \mapsto (\omega^{-1}(e_1 - e_2), \omega^{-1}(e_1 - e_2)).$$

5.3. **Comparison.**

Theorem 5.3. *There is an equivalence* $\mathcal{L}(1) \circledcirc \mathcal{L}(1) \xrightarrow{\sim} \mathscr{T}$ *of 2-representations.*

We will use a few intermediate steps.

Define a new algebra R:

$$R = P_2[e] / (e^2 - \omega e).$$

There is a map of P_2-algebras $R \xrightarrow{\gamma} B_{s_1}$ given by $e \mapsto 1 \otimes y_1 - y_1 \otimes 1$. There is another map of P_2-algebras $R \xrightarrow{\gamma'} Q_1^{\mathsf{op}}$ given by $P_2 \ni f \mapsto (f, f) \in Q_1^{\mathsf{op}}$ and $e \mapsto (\omega, 0)$.

Lemma 5.4. *The maps γ and γ' are isomorphisms of P_2-algebras.*

Proof. Straightforward. □

We will also use the composition $\sigma \circ \gamma'$ to obtain an isomorphism of (P_2, P_2)-bimodules $R \xrightarrow{\sim} Q_2$ given by $f \mapsto (f, f)$ and $e \mapsto (0, \omega)$.

Now we translate $\mathscr{T}$ using γ. The action of B_{s_1} on P_2 induces an action of R on P_2 through γ, according to which $P_2 \hookrightarrow R$ acts on P_2 by multiplication, and e acts by zero. We have an isomorphism of R-modules $P_2 \xrightarrow{\sim} R/(e)$ using this action on P_2. In the remainder of this section we assume this isomorphism and write R in place of B_{s_1} everywhere in the 2-representation $\mathscr{T}$. Under this translation, and using the decomposition $R \xrightarrow{\sim} P_2 \oplus P_2e$ as P_2-modules, we have:

$$_{+2}x_0 = \begin{pmatrix} y_2 & y_2 + e \end{pmatrix}, \quad _0x_{-2} = \begin{pmatrix} y_1 \\ y_1 - e \end{pmatrix},$$

and

$$_{+2}\tau_{-2} = \big(p_1 + p_2e \mapsto -p_2\big).$$

Lemma 5.5. *The matrix presentation of T_0 is given by:*

$$\begin{pmatrix} P_2 & P_2 \\ P_2 & R \end{pmatrix} \xrightarrow{\sim} T_0,$$

where:

- *for $[T_0]_{11}$ the map sends $\theta \in P_2$ to $(1 \mapsto \theta) \in \mathrm{End}_R(P_2)^{\mathsf{op}}$*
- *for $[T_0]_{21}$ the map sends $\theta \in P_2$ to $(1 \mapsto \theta) \in \mathrm{Hom}_R(R, P_2)$*
- *for $[T_0]_{12}$ the map sends $\theta \in P_2$ to $(1 \mapsto \theta\omega - \theta e) \in \mathrm{Hom}_R(P_2, R)$*
- *for $[T_0]_{22}$ the map sends $r \in R$ to $(1 \mapsto r) \in \mathrm{End}_R(R, R)^{\mathsf{op}}$.*

The algebra structure maps (cf. §2.4) are given as follows:

- *$[T_0]_{11} \circlearrowright [T_0]_{12}$ by $\theta.\theta' = \theta\theta'$*
- *$[T_0]_{21} \circlearrowleft [T_0]_{11}$ by $\theta'.\theta = \theta'\theta$*
- *$[T_0]_{12} \circlearrowleft [T_0]_{22}$ by $\theta.(p_1 + p_2e) = \theta p_1$*
- *$[T_0]_{22} \circlearrowright [T_0]_{21}$ by $(p_1 + p_2e).\theta = p_1\theta$*
- *$[T_0]_{12} \otimes [T_0]_{21} \to [T_0]_{11}$ by $\theta \otimes \theta' \mapsto \omega\theta\theta'$*
- *$[T_0]_{21} \otimes [T_0]_{12} \to [T_0]_{22}$ by $\theta' \otimes \theta \mapsto \omega\theta'\theta - \theta'\theta e$.*

Proof. Let us explain the map to $[T_0]_{12}$. Recall that $P_2 \cong R/(e)$. An element of $\mathrm{Hom}_R(R/(e), R)$ is given by the image $r = p_1 + p_2e$ of 1, which may be anything satisfying $e.r = 0$, and that condition is equivalent to $p_1 = -p_2\omega$. The other morphisms and the structure maps are easily computed. □

Lemma 5.6. *Let $\Phi_0 : T_0 \to C_0$ be given on components by:*

$$\begin{pmatrix} Id_{P_2} & \omega \\ Id_{P_2} & \gamma' \end{pmatrix}.$$

Then Φ_0 is an isomorphism of P_2-algebras.

Proof. The specified maps give algebra isomorphisms on the diagonal components, and k-module isomorphisms on the off-diagonal components. Now we check equivariance under the bimodule structure maps. The only nonobvious cases concern maps involving the lower right component.

An element of Q_1^{op} may be written uniquely as a sum $(\omega\theta, 0) + (\varphi, \varphi)$. This is sent by γ'^{-1} to $\varphi + \theta e \in R$. So the action of (θ, φ) by multiplication by φ agrees with the action of $p_1 + p_2e$ by multiplication by p_1. The structure map $[T_0]_{12} \otimes [T_0]_{21} \to [T_0]_{11}$ clearly agrees with Eq. 5.1 through Φ_0. The map $[T_0]_{21} \otimes [T_0]_{12} \to [T_0]_{22}$ agrees with Eq. 5.2 through Φ_0 because $\gamma' : \omega\theta'\theta - \theta'\theta e \mapsto (0, \omega\theta\theta')$. □

Proof of Theorem 5.3. Extend Φ_0 to an algebra isomorphism $\Phi : T \xrightarrow{\sim} C$ by $\Phi_{+2} = \mathrm{Id}_{P_2}$ and $\Phi_{-2} = \mathrm{Id}_{P_2}$. It remains to check compatibility with the actions of E, x, and τ in $\mathcal{U}^+$, and this poses no difficulty. We summarize that now.

We have in $\mathscr{T}$ that ${}_0\mathscr{E}_{-2} \xrightarrow{\sim} T_0 r_2$ for $r_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in [T_0]$, and similarly ${}_0\tilde{E}_{-2} = C_0 q_2$ in $\mathcal{L}(1) \bigotimes \mathcal{L}(1)$; and we have $q_2 = \Phi_0(r_2)$. The action of ${}_0x_{-2}$ on ${}_0\mathscr{E}_{-2}$ in $\mathscr{T}$ can be written in $T_0 r_2$ as multiplication on the left by $\left(\begin{smallmatrix} y_1 & 0 \\ 0 & y_1 - e \end{smallmatrix}\right) \in [T_0]$. In $\mathcal{L}(1) \bigotimes \mathcal{L}(1)$ it is written as multiplication on the left by $\left(\begin{smallmatrix} y_1 & 0 \\ 0 & (y_2, y_1) \end{smallmatrix}\right)$. These correspond using $\gamma' : R \xrightarrow{\sim} Q_1^{\mathsf{op}}$. Similarly for ${}_{+2}x_0$ since $\gamma' : R \ni y_2 + e \mapsto (y_1, y_2) \in Q_1^{\mathsf{op}}$. Finally, the action of τ in R by ${}_{+2}\tau_{-2} = \big(p_1 + p_2 e \mapsto -p_2\big)$ corresponds to ${}_{+2}\tilde{\tau}_{-2}$, now using $\sigma \circ \gamma' : R \xrightarrow{\sim} Q_2$. □

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