GENERALIZATIONS OF PAPPUS' CENTROID THEOREM VIA STOKES' THEOREM

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Abstract. This paper provides a novel proof of a generalization of Pappus' Centroid Theorem on n-dimensional tubes using Stokes' Theorem on manifolds.

1. Introduction

The (second) Pappus Centroid Theorem or the Pappus-Guldin Theorem states that the volume of a solid of revolution generated by rotating a plane region $\mathcal{R}$ with piecewise-smooth boundary about an axis L is $2\pi r \cdot \text{Area}(\mathcal{R})$, where r is the distance from the centroid of $\mathcal{R}$ to L. This result generalizes considerably to the following main theorem.

Theorem 1.1 (Main Theorem)**.** *Let C be a simple, regular, smooth curve in $\mathbb{R}^n$. Let $\mathcal{R}$ be a region in $\mathbb{R}^{n-1}$ whose boundary is an embedding of the $(n-2)$-dimensional sphere $\mathbb{S}^{n-2}$. Let $\mathcal{W}$ be a region in $\mathbb{R}^n$ whose boundary is a generalized tube around C such that the cross-section normal to C of $\mathcal{W}$ at each point P of C is the region $\mathcal{R}$ with centroid at P. Assuming the cross-section $\mathcal{R}$ rotates smoothly as it "travels" along C, then*

$$\text{Vol}_n(\mathcal{W}) = \text{length}(C) \cdot \text{Vol}_{n-1}(\mathcal{R}).$$

The Pappus Centroid Theorem follows from this Main Theorem by taking $n = 3$, C to be a circle in $\mathbb{R}^3$, and $\mathcal{R}$ to remain fixed with respect to the principal normal to C in the normal plane. This theorem recently was proved by Gray, Miquel, and Domingo-Juan in [1] and [3] using parallel transport. However, in 1959 in [2], these authors proved this theorem in a special case for $\mathbb{R}^3$ using elementary methods related to Stokes' Theorem. This article proves the Main Theorem in full generality using Stokes' Theorem on manifolds. In this regard, we can consider the proof elementary compared to those in [1] and [3].

Before proving the Main Theorem in full generality, we sketch Goodman's proof of it in $\mathbb{R}^3$, leaving the reader to consult [2] for details. The description of the generalized tube and the method involving the divergence theorem motivate the situation for arbitrary n.

Date: November 25, 2014.

2010 *Mathematics Subject Classification.* Primary 58C00; Secondary 58C35, 53A07.

The authors thank Wheaton College's Faculty Student Mentoring Initiative, which made possible the collaboration that led to this article.

2. Generalized Tubes In Dimension 3

Definition 2.1. Let C be a simple, regular, smooth space curve and let $\mathcal{R}$ be a compact planar region with one boundary component $\partial\mathcal{R}$ is a piecewise smooth simple closed curve. Select a marked point P in $\mathcal{R}$. C has a normal plane at each point. Let $\mathcal{W}$ be a region in $\mathbb{R}^3$ such that the intersection of $\mathcal{W}$ with the normal plane to C at any point is isometric to the region $\mathcal{R}$, with the corresponding marked point P lying on the curve C. We assume $\mathcal{R}$ rotates smoothly in the normal plane to C as it travels along C. Such a region $\mathcal{W}$ is called a *generalized tube* along C with cross-section $\mathcal{R}$ and center P.

This definition allows for rotational freedom of $\mathcal{R}$ around the marked point P in the normal planes to C. However, this rotational varies smoothly. We may also describe the generalized tube as a fiber-bundle over C with fiber $\mathcal{R}$, that is a subbundle of the normal bundle over C.

Figure 1 depicts two generalized tubes around a portion of a helix. More precisely, the figure depicts the tube boundary excluding the "caps," or cross-sections at the end points of C. The planar curve shows its generating region $\mathcal{R}$ where the marked point of $\mathcal{R}$ is the origin.

Let $\mathcal{S}$ be the boundary $\partial\mathcal{W}$ of a generalized tube excluding the caps. (If C is a closed curve, then $\partial\mathcal{W}$ has no caps.) Suppose that $\alpha : [0,\ell] \to \mathbb{R}^3$ gives a parametrization by arclength of C. Also suppose that $\vec{\beta} : [0,c] \to \mathbb{R}^2$ is a parametrization of $\partial\mathcal{R}$ placing the marked point P at the origin. We write $\vec{\beta}(u) = \big(x(u), y(u)\big)$ for the coordinate functions. A parametrization for $\mathcal{S}$ is

$$\begin{aligned}\vec{X}(s,u) = \vec{\alpha}(s) &+ (\cos(\theta(s))x(u) - \sin(\theta(s))y(u))\vec{P}(s) \\ &+ (\sin(\theta(s))x(u) + \cos(\theta(s))y(u))\vec{B}(s),\end{aligned} \tag{1}$$

for some function $\theta(s)$, where $\vec{P}(s)$ and $\vec{B}(s)$ are respectively the principal normal and binormal vector functions to $\vec{\alpha}(s)$.

Recall that $(\vec{T}(s), \vec{P}(s), \vec{B}(s))$, where $\vec{T}$, $\vec{P}$, and $\vec{B}$ are the usual tangent, principal normal, and binormal vectors to $\vec{\alpha}(s)$, is called the Frenet frame to $\vec{\alpha}(s)$. The function $\theta(s)$ determines the rotation of the region $\mathcal{R}$ around the origin with respect to the Frenet frame. The stipulation that $\mathcal{R}$ rotates smoothly as it moves along C implies that $\theta(s)$ is a smooth function.

Figure 1(b) depicts a generalized tube where the x-axis in the depiction of $\mathcal{R}$ always lies along the principal normal vector of $\vec{\alpha}(s)$, and Figure 1(c) depicts a generalized tube with the same cross-section region but having some rotation with respect to the basis $(\vec{P}, \vec{B})$ in the normal plane. For brevity, we write

$$\vec{X} = \vec{\alpha} + (x\cos\theta - y\sin\theta)\vec{P} + (x\sin\theta + y\cos\theta)\vec{B}$$

where functional dependence is understood from (1).

Theorem 2.2 (Corollary 2 in [2])**.** *The volume of a generalized tube as described in Definition 2.1 is* $V = \text{length}(C) \cdot \text{Area}(\mathcal{R})$.

Goodman's method to calculate the volume uses the fact that the position vector field $\vec{r}(x,y,z) = (x,y,z)$ has divergence everywhere 3. So, using the notation defined

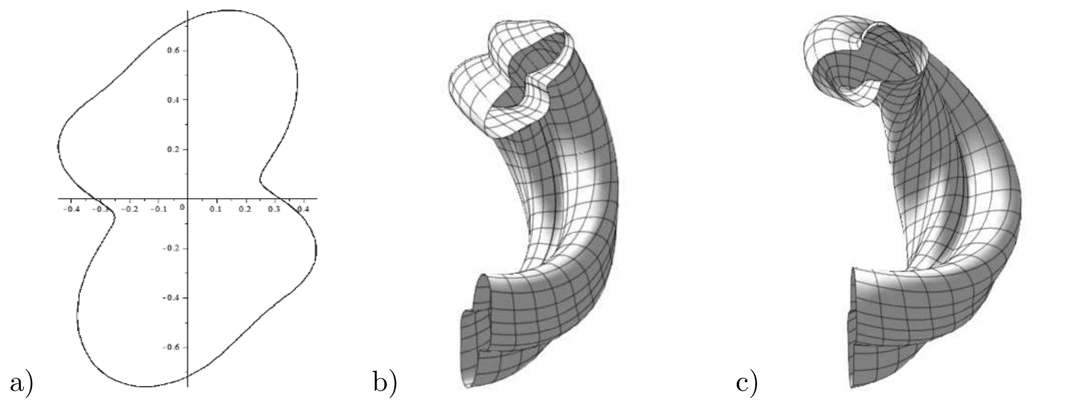

FIGURE 1. A generalized tube with its generating region.

above, the volume of the generalized tube is

$$\mathrm{Vol}(\mathcal{W}) = \frac{1}{3}\iiint_{\mathcal{W}} 3\,dV = \frac{1}{3}\iiint_{\mathcal{W}} \nabla\cdot\vec{r}\,dV = \frac{1}{3}\iint_{\partial\mathcal{W}} \vec{r}\cdot d\vec{A}$$

where $d\vec{A}$ is the outward pointing surface element. Note that $\partial\mathcal{W}$ consists of the tube's outward surface $\mathcal{S}$, parametrized by $\vec{X}$, and the end caps (if C is not a closed curve). Over $\mathcal{S}$, $d\vec{A}$ is given by $d\vec{A} = (\vec{X}_u \times \vec{X}_s)\,du\,ds$ with $(u,v)\in[0,c]\times[0,\ell]$ while on the end caps, $d\vec{A} = -\vec{T}(0)dA$ when $s=0$ and $d\vec{A} = \vec{T}(\ell)dA$ when $s=\ell$. The caps, like any cross-section at s, are parametrized by

$$\vec{Y}_s(p,q) = \vec{\alpha}(s) + p\vec{P}(s) + q\vec{B}(s) \qquad \text{for } (p,q)\in\mathcal{R}_s,$$

where $\mathcal{R}_s$ is the region $\mathcal{R}$ rotated about the origin (the marked point P) by the angle $\theta(s)$. Thus, since $\vec{T}(s)$ is perpendicular to both $\vec{P}(s)$ and $\vec{B}(s)$, we have

$$\begin{aligned}
&3\,\mathrm{Vol}(\mathcal{W})\\
&\quad= \int\limits_{s=0}^{\ell}\int\limits_{u=0}^{c} \vec{X}\cdot(\vec{X}_u\times\vec{X}_s)\,du\,ds + \iint\limits_{\mathcal{R}_\ell} \vec{\alpha}(\ell)\cdot\vec{T}(\ell)\,dp\,dq + \iint\limits_{\mathcal{R}_0} -\vec{\alpha}(0)\cdot\vec{T}(0)\,dp\,dq\\
&\quad= \int\limits_{s=0}^{\ell}\int\limits_{u=0}^{c} \vec{X}\cdot(\vec{X}_u\times\vec{X}_s)\,du\,ds + \mathrm{Area}(\mathcal{R})\left(\vec{\alpha}(\ell)\cdot\vec{T}(\ell) - \vec{\alpha}(0)\cdot\vec{T}(0)\right).
\end{aligned}\tag{2}$$

The problem of calculating the volume of $\mathcal{W}$ reduces to calculating the double integral in (2).

Recall that vectors of the Frenet frame (parametrized by arclength) differentiate according to

$$\begin{array}{rcrcr}
\vec{T}' & = & & \kappa\vec{P} & \\
\vec{P}' & = & -\kappa\vec{T} & & \tau\vec{B}\\
\vec{B}' & = & & -\tau\vec{P}, &
\end{array}\tag{3}$$

where $\kappa(s)$ and $\tau(s)$ are the curvature and torsion functions of the space curve $\vec{\alpha}(s)$. Then the tangent vectors to $\vec{X}$ are given (after simplification) by

$$\begin{aligned}
\vec{X}_u &= (x'\cos\theta - y'\sin\theta)\vec{P} + (x'\sin\theta + y'\cos\theta)\vec{B} \\
\vec{X}_s &= (1 - \kappa x\cos\theta + \kappa y\sin\theta)\vec{T} - (\theta' + \tau)(x\sin\theta + y\cos\theta)\vec{P} \\
&\qquad + (\theta' + \tau)(x\cos\theta - y\sin\theta)\vec{B}.
\end{aligned}$$

So

$$\begin{aligned}
\vec{X}_u \times \vec{X}_s = (\theta' + \tau)(xx' + yy')\vec{T} &+ (1 - \kappa x\cos\theta + \kappa y\sin\theta)(x'\sin\theta + y'\cos\theta)\vec{P} \\
&- (1 - \kappa x\cos\theta + \kappa y\sin\theta)(x'\cos\theta - y'\sin\theta)\vec{B}.
\end{aligned}$$

The dot product $\vec{X}\cdot(\vec{X}_u \times \vec{X}_s)$ involves many terms. However, all of the additive terms involved in the integrals are multiplicatively separable, which, by the usual corollary to Fubini's Theorem, allows us to separate the double integral. Many of the integrals involving u vanish or evaluate to a simple constant, namely the area of the cross-section. Consider the following integrals. By substitution,

$$\int\limits_{u=0}^{c} xx'\,du = x^2\Big|_0^c = 0$$

because $(x(u), y(u))$ with $u \in [0, c]$ parametrizes a closed curve $\partial\mathcal{R}$. By similar reasoning, the following integrals are all 0:

$$\int\limits_{u=0}^{c} x'\,du = 0, \qquad \int\limits_{u=0}^{c} y'\,du = 0, \qquad \int\limits_{u=0}^{c} xx'\,du = 0, \qquad \int\limits_{u=0}^{c} yy'\,du = 0. \tag{4}$$

By Green's Theorem for the area of the interior of a simple closed piecewise smooth curve,

$$\int_{u=0}^{c} xy'\,du = -\int_{u=0}^{c} yx'\,du = \iint\limits_{\mathcal{R}} 1\,dA = \mathrm{Area}(\mathcal{R}). \tag{5}$$

Also by Green's Theorem,

$$\int_{u=0}^{c} \frac{1}{2}x^2y'\,du = \int_{u=0}^{c} -xyx'\,du = \iint\limits_{\mathcal{R}} x\,dA = 0 \tag{6}$$

because this integral is the y-moment of $\mathcal{R}_s$ and by hypothesis the centroid of $\mathcal{R}_s$ is $(0,0)$ for all s. By the same reasoning but for the x-moment, we also have

$$\int_{u=0}^{c} -\frac{1}{2}y^2x'\,du = \int_{u=0}^{c} xyy'\,du = \iint\limits_{\mathcal{R}} y\,dA = 0. \tag{7}$$

Upon applying these integrals only a few terms remain in (2). Setting $A = \mathrm{Area}(\mathcal{R})$, we get

$$3\,\mathrm{Vol}(\mathcal{W}) = \int\limits_{s=0}^{\ell} (-\vec{\alpha}\cdot\vec{T}'A + 2A)\,ds + A\left(\vec{\alpha}(\ell)\cdot\vec{T}(\ell) - \vec{\alpha}(0)\cdot\vec{T}(0)\right).$$

Using integration by parts on the dot product, we obtain

$$\begin{aligned}
3 \cdot \mathrm{Vol}(\mathcal{W}) &= -A \left(\vec{\alpha} \cdot \vec{T}\right)\Big|_0^\ell + A \int\limits_{s=0}^{\ell} \vec{\alpha}' \cdot \vec{T}\, ds + 2A\ell + A \left(\vec{\alpha}(\ell) \cdot \vec{T}(\ell) - \vec{\alpha}(0) \cdot \vec{T}(0)\right) \\
&= -A \left(\vec{\alpha}(\ell) \cdot \vec{T}(\ell) - \vec{\alpha}(0) \cdot \vec{T}(0)\right) \\
&\qquad + A \int\limits_{s=0}^{\ell} \vec{T} \cdot \vec{T}\, ds + 2A\ell + A \left(\vec{\alpha}(\ell) \cdot \vec{T}(\ell) - \vec{\alpha}(0) \cdot \vec{T}(0)\right) \\
&= A\ell + 2A\ell = 3A\ell.
\end{aligned}$$

We conclude that $\mathrm{Vol}(\mathcal{W}) = \mathrm{Area}(\mathcal{R}) \cdot \mathrm{length}(C)$.

Theorem 2.2 establishes the Main Theorem of the paper for generalized tubes in $\mathbb{R}^3$. In order to prove the Main Theorem in full generality, we will need to use differential forms along with Stokes' Theorem on manifolds. However, a key component to the Main Theorem is a set of integral formulas for the general case similar to (4), (5), (6), and (7).

3. Volumes, Moments, and Zero Integrals for Solids in $\mathbb{R}^m$

Recall that Stokes' Theorem on manifolds states that if M is an m-dimensional, oriented manifold with boundary ∂M, and ω is a differential $(m-1)$-form on M, then

$$\int_{\partial M} \omega = \int_M d\omega, \tag{8}$$

where ∂M has the boundary orientation inherited from the orientation on M.

Definition 3.1. We define a *solid* in $\mathbb{R}^m$ as a compact embedded m-dimensional submanifold of $\mathbb{R}^m$ with boundary ∂M. We assume the pull-back orientation on M.

We define the $(m-1)$-form η^i in $\mathbb{R}^m$ by

$$\eta^i = (-1)^{i+1} dy^1 \wedge dy^2 \wedge \cdots \wedge \widehat{dy^i} \wedge \cdots \wedge dy^m$$

where $(y^1, y^2, \ldots, y^m)$ is a coordinate system on $\mathbb{R}^m$ and $\widehat{\ }$ denotes removal of that term.

Lemma 3.2. *The m-dimensional volume of a solid M is*

$$\mathrm{Vol}_m(M) = \int_{\partial M} y^i \eta^i$$

for any $i = 1, 2, \ldots, m$.

Proof. The differential of $y^i \eta^i$ is

$$d(y^i\eta^i) = (-1)^{i+1} dy^i \wedge dy^1 \wedge dy^2 \wedge \cdots \wedge \widehat{dy^i} \wedge \cdots \wedge dy^m = dy^1 \wedge dy^2 \wedge \cdots \wedge dy^m.$$

This form is precisely the volume form on $\mathbb{R}^m$ and thus on the solid M as well. Hence, by Stokes' Theorem,

$$\int_{\partial M} y^i \eta^i = \int_M dy^1 \wedge dy^2 \wedge \cdots \wedge dy^m = \mathrm{Vol}_m(M). \qquad \square$$

This lemma immediately implies the following corollary:

Corollary 3.3. *Let* $\nu = \dfrac{1}{m}\sum_{i=1}^{m} y^i \eta^i$. *The* m*-dimensional volume of* M *is*

$$\mathrm{Vol}_m(M) = \int_{\partial M} \nu.$$

In this article, if $F : M \to N$ is a differentiable map between differentiable manifolds, we will denote by $[dF]$ the matrix of functions of the differential dF in reference to given coordinate systems on M and on N. Furthermore, when the dimension of M is one less than the dimension of N and when coordinate systems on neighborhoods of M and N are implied, we denote by $|d_j F|$ the determinant of $[dF]$ in which the jth row is removed.

Proposition 3.4. *Let* M *be an* m*-dimensional solid such that the boundary* ∂M *is the embedding of a continuous map* $H : \mathbb{S}^{m-1} \to \mathbb{R}^m$ *that is smooth except on a subset of measure 0 in* $\mathbb{S}^{m-1}$. *Suppose also that* H *induces an orientation on* ∂M *that is compatible with the boundary orientation induced from* M. *Let* ν *be the* $(m-1)$*-form as in Corollary 3.3 and let* ω *be the* $(m-1)$*-form on* $\mathbb{S}^{m-1}$ *given by* $\omega = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^{m-1}$ *for coordinates* $(x^1, x^2, \ldots, x^{m-1})$. *The* m*-dimensional volume of* M *is*

$$\mathrm{Vol}_m(M) = \int_{H(\mathbb{S}^{m-1})} \nu = \int_{\mathbb{S}^{m-1}} H^*\nu = \frac{1}{m}\int_{\mathbb{S}^{m-1}} \det\big(H,\, [dH]\big)\, \omega, \tag{9}$$

where in $\det\big(H,\, [dH]\big)$ *we write the components of* H *as a column vector. If* H *induces the opposite orientation, the second two integrals change sign.*

Proof. The equality

$$\mathrm{Vol}_m(M) = \int_{H(\mathbb{S}^{m-1})} \nu$$

follows immediately from Corollary 3.3. Let $(x^1, x^2, \ldots, x^{m-1})$ be coordinates on $\mathbb{S}^{m-1}$ and $(y^1, y^2, \ldots, y^m)$ on $\mathbb{R}^m$. Notice that the pullback of ν by H is

$$H^*\nu = \frac{1}{m}\sum_{i=1}^{m} H^i (-1)^{i+1}\, dH^1 \wedge dH^2 \wedge \cdots \wedge \widehat{dH^i} \wedge \cdots \wedge dH^m.$$

Or, writing in x^i coordinates, and using the fact that

$$dH^i = \frac{\partial H^i}{\partial x^j} dx^j$$

(assuming the Einstein summation convention), we find

$$H^*\nu = \frac{1}{m}\sum_{i=1}^{m} H^i (-1)^{i+1} \left(\frac{\partial H^1}{\partial x^{j_1}} dx^{j_1}\right) \wedge \left(\frac{\partial H^2}{\partial x^{j_2}} dx^{j_2}\right) \wedge \cdots \wedge \widehat{\left(\frac{\partial H^i}{\partial x^{j_i}} dx^{j_i}\right)} \wedge \cdots \wedge \left(\frac{\partial H^m}{\partial x^{j_m}} dx^{j_m}\right).$$

By Theorem C.5.22 in [4], this is equivalent to

$$H^*\nu = \frac{1}{m}\sum_{i=1}^{m} H^i(-1)^{i+1} \begin{vmatrix} \frac{\partial H^1}{\partial x^1} & \frac{\partial H^1}{\partial x^2} & \cdots & \frac{\partial H^1}{\partial x^{m-1}} \\ \frac{\partial H^2}{\partial x^1} & \frac{\partial H^2}{\partial x^2} & \cdots & \frac{\partial H^2}{\partial x^{m-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{\frac{\partial H^i}{\partial x^1}} & \widehat{\frac{\partial H^i}{\partial x^2}} & \cdots & \widehat{\frac{\partial H^i}{\partial x^{m-1}}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial H^m}{\partial x^1} & \frac{\partial H^m}{\partial x^2} & \cdots & \frac{\partial H^m}{\partial x^{m-1}} \end{vmatrix} dx^1 \wedge dx^2 \wedge \cdots \wedge dx^{m-1}.$$

Taking the summation and recognizing the Laplace expansion of a determinant down the first column, we see that

$$H^*\nu = \frac{1}{m}\det\big(H,\, [dH]\big)\; dx^1 \wedge dx^2 \wedge \cdots \wedge dx^{m-1}.$$

Then (9) follows. Note that the second integral changes sign if H induces the opposite orientation on ∂M, so the third integral changes sign as well. □

Lemma 3.2, Corollary 3.3, and Proposition 3.4 are generalizations to higher dimensions of Green's Theorem for area. For example, suppose that $\mathcal{S}$ is a solid in $\mathbb{R}^3$ such that the boundary $\partial\mathcal{S}$ is parametrized by $\vec{X}(u,v) = (x(u,v), y(u,v), z(u,v))$ with $(u,v) \in \mathcal{D}$ such that $\vec{X}_u \times \vec{X}_v$ is outward-pointing. Then by Proposition 3.4, the volume of $\mathcal{S}$ is

$$\mathrm{Vol}(\mathcal{S}) = \frac{1}{3}\iint_{\mathcal{D}} \begin{vmatrix} x & x_u & x_v \\ y & y_u & y_v \\ z & z_u & z_v \end{vmatrix}\, du\, dv.$$

Because of the flexibility in Stokes' Theorem, as in Green's Area Theorem, this formula still applies when $\partial\mathcal{S}$ is piecewise smooth. In that case, we interpret the above integral as a sum of integrals taken over domains $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_r$ such that the parametrizations for the smooth pieces of $\partial\mathcal{S}$ have domains $\mathcal{D}_i$. The same principle applies in (9).

We will encounter other integrals that cancel. We list them here.

Proposition 3.5. *Let M be a solid and let $(y^1, y^2, \ldots, y^m)$ be a coordinate system on M. Then for i and q in $\{1, 2, \ldots, m\}$,*

$$\int_{\partial M} y^q \eta^i = \delta_i^q\, \mathrm{Vol}_m(M),$$

where δ_i^q is the Dirac delta in which $\delta_i^q = 1$ if $i = q$ and $\delta_i^q = 0$ if $i \neq q$.

Proof. The case with $i = q$ is Lemma 3.2. If $i \neq q$, then

$$d(y^q\eta^i) = dy^q \wedge dy^1 \wedge dy^2 \wedge \cdots \wedge \widehat{dy^i} \wedge \cdots \wedge dy^m = 0$$

because one differential is repeated. Then by Stokes' Theorem, we have

$$\int_{\partial M} y^q\eta^i = \int_M d(y^q\eta^i) = \int_M 0 = 0.$$ □

Corollary 3.6. *Let M, H, and ω be as in Proposition 3.4. Then*

$$\int_{\mathbb{S}^{m-1}} (-1)^{i+1} H^q |d_i H| \omega = \delta_i^q \operatorname{Vol}_m(M).$$

Proof. This follows immediately from the fact that $(-1)^{i+1} H^q |d_i H| \omega = H^*(y^q \eta^i)$. □

Proposition 3.7. *Let M, H, and ω be as in Proposition 3.4. Let $\vec{a} = (a^1, a^2, \dots, a^m)$ be a constant vector, listed as a column vector. Then*

$$\int_{\mathbb{S}^{m-1}} \det(\vec{a}, [dH]) \omega = 0.$$

Proof. By the reasoning in the proof of Proposition 9, we see that

$$\begin{aligned} \det(\vec{a}, [dH]) \omega &= \sum_{i=1}^{m} (-1)^{i+1} a^i dH^1 \wedge dH^2 \wedge \cdots \wedge \widehat{dH^i} \wedge \cdots \wedge dH^m \\ &= H^* \left(\sum_{i=1}^{m} (-1)^{i+1} a^i dy^1 \wedge dy^2 \wedge \cdots \wedge \widehat{dy^i} \wedge \cdots \wedge dy^m \right). \end{aligned}$$

Hence, by a pull-back and then Stokes' Theorem,

$$\begin{aligned} \int_{\mathbb{S}^{m-1}} \det(\vec{a}, [dH]) &= \int_{H(\mathbb{S}^{m-1})} \sum_{i=1}^{m} (-1)^{i+1} a^i dy^1 \wedge dy^2 \wedge \cdots \wedge \widehat{dy^i} \wedge \cdots \wedge dy^m \\ &= \int_M d \left(\sum_{i=1}^{m} (-1)^{i+1} a^i dy^1 \wedge dy^2 \wedge \cdots \wedge \widehat{dy^i} \wedge \cdots \wedge dy^m \right) \\ &= \int_M 0 = 0. \end{aligned}$$

□

In the proof of Theorem 2.2, certain integrals vanished by virtue of the cross-section always having its centroid on the curve C, and the same thing occurs in higher dimensions. The following proposition establishes the centroid generalizations needed later:

Proposition 3.8. *Let M be an m-dimensional solid as given in Definition 3.1. Let $(y^1, y^2, \dots, y^m)$ be a coordinate system covering M. Let $(\bar{y}^1, \bar{y}^2, \dots, \bar{y}^m)$ be the center of mass of M. Then*

$$\int_{\partial M} y^p y^q \eta^i = \begin{cases} 0 & \text{if } p \neq i \text{ and } q \neq i \\ \bar{y}^p \operatorname{Vol}_m(M) & \text{if } p \neq i \text{ and } q = i \\ 2\bar{y}^i \operatorname{Vol}_m(M) & \text{if } p = q = i. \end{cases}$$

Proof. By Stokes' Theorem,

$$\int_{\partial M} y^p y^q \eta^i = \int_M d(y^p y^q \eta^i).$$

However,

$$d(y^p y^q \eta^i) = (y^q dy^p + y^p dy^q) \wedge \eta^i = y^q dy^p \wedge \eta^i + y^p dy^q \wedge \eta^i.$$

If neither $p = i$ nor $q = i$, then $dy^p \wedge \eta^i = 0$ and $dy^q \wedge \eta^i = 0$. If $q = i$ and $p \neq i$, then $d(y^p y^q \eta^i) = y^p \, dy^1 \wedge dy^2 \wedge \cdots \wedge dy^m$ and

$$\int_M y^p \, dy^1 \wedge dy^2 \wedge \cdots \wedge dy^m = \bar{y}^p \operatorname{Vol}_m(M)$$

by definition of the center of mass. Finally, if $p = q = i$, then $d(y^p y^q \eta^i) = 2y^i \, dy^1 \wedge dy^2 \wedge \cdots \wedge dy^m$ and

$$\int_M 2y^i \, dy^1 \wedge dy^2 \wedge \cdots \wedge dy^m = 2\bar{y}^i \operatorname{Vol}_m(M). \qquad \square$$

4. Generalized Tubes In Higher Dimensions

We are finally in position to prove the Main Theorem 1.1.

We must first set up a useful description of a generalized tube. Let $\mathcal{W}$ be a generalized tube with guiding curve C and cross-section $\mathcal{R}$ as described in the statement of the Main Theorem. A generalized tube is a fiber-bundle over C with fiber $\mathcal{R}$, that is a subbundle of the normal bundle over C. Suppose that C is parametrized by arclength by $\alpha : [0, \ell] \to \mathbb{R}^n$. Suppose that the cross-section $\mathcal{R}$ is a solid in $\mathbb{R}^{n-1}$ whose boundary $\partial\mathcal{R}$ is parametrized by an orientation-preserving, differentiable map $H : \mathbb{S}^{n-2} \to \mathbb{R}^{n-1}$. We also assume that $\mathcal{R}$ rotates smoothly about the origin in the normal plane as it is transported along C. For the purpose of the theorem, we also assume that the center of mass of $\mathcal{R}$ is the origin in $\mathbb{R}^{n-1}$. Define $\bar{H} : \mathbb{S}^{n-2} \to \mathbb{R}^n$ by $\bar{H}(\vec{x}) = (0, H(\vec{x}))$.

The boundary $\partial\mathcal{W}$ of the solid generalized tube consists of the caps at $\alpha(0)$ and $\alpha(\ell)$ as well as the side surface $\mathcal{S}$, which we can parametrize by

$$\alpha(t) + M(t)\bar{H}(\vec{x}) \qquad \text{for } (t, \vec{x}) \in [0, \ell] \times \mathbb{S}^{n-2},$$

where $M : [0, \ell] \to \mathrm{SO}(n)$ is a differentiable curve of special orthogonal (rotation) matrices in $\mathbb{R}^n$ such that for all t,

$$M(t) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = M(t)\vec{e}_1 = \alpha'(t). \tag{10}$$

Note that since $M(t)$ is a rotation matrix and the unit vector $\vec{e}_1$ in the y^1 direction is perpendicular to $\{(0, y^2, \ldots, y^n) \,|\, y^i \in \mathbb{R}\}$, then for all $t \in [0, \ell]$, the boundary of the cross-section $M(t)\bar{H}(\vec{x})$, for $\vec{x} \in \mathbb{S}^{n-2}$, is in a plane perpendicular to the tangent vector $\alpha'(t)$. For simplicity later, we write $F(t, \vec{x}) = M(t)\bar{H}(\vec{x})$.

Recall that since $M(t)$ is a special orthogonal matrix for all t, then $M(t)^{-1} = M(t)^\top$, $\det M(t) = 1$, and $M'(t) = M(t)A(t)$, where $A(t)$ is some anti-symmetric matrix for all t. Using the rotation matrix $M(t)$ provides the following useful fact.

Lemma 4.1. *The first component of the vector $M(t)^{-1}\alpha(t)$ is equal to the dot product $\alpha(t) \cdot \alpha'(t)$.*

Proof. By (10), the dot product $\alpha(t) \cdot \alpha'(t)$ is

$$\alpha(t) \cdot \alpha'(t) = \alpha(t)^\top M(t)\vec{e}_1.$$

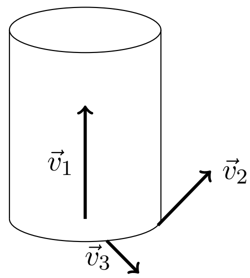

FIGURE 2. Reversed orientation on a cylinder

Taking the transpose of the matrix expression on the right, and since the whole expression is just a real number, we get

$$\alpha(t) \cdot \alpha'(t) = \vec{e}_1^{\top} M(t)^{\top} \alpha(t) = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix} M(t)^{-1} \alpha(t)$$

and the lemma follows. □

Proof of the Main Theorem 1.1. **Case 1:** Assume that the guiding curve C is not closed. Let ν be the $(n-1)$-form $\nu = \frac{1}{n} \sum_{i=1}^{n} y^i \eta^i$. By Corollary 3.3, the volume of the generalized tube is

$$\mathrm{Vol}_n(\mathcal{W}) = \int_{\partial \mathcal{W}} \nu = \int_{\mathcal{S}} \nu + \int_{\mathrm{cap}_{t=0}} \nu + \int_{\mathrm{cap}_{t=\ell}} \nu. \tag{11}$$

We parametrize $\mathcal{S}$ by $\alpha + F$ but we note that this parametrization is orientation-reversing. This can be seen by applying our setup to the case of a circular cylinder in $\mathbb{R}^3$ and generalizing to higher dimensions. In Figure 2, $\vec{v}_1$ is $\alpha'(t)$, $\vec{v}_2$ is a tangent vector to the cross-section boundary in positive orientation, and $\vec{v}_3$ is the outward pointing normal vector to the solid M. These three vectors form a left-handed system so the orientation induced from our parametrization is reversed from the boundary orientation on ∂M induced by the standard orientation of $\mathbb{R}^n$ on M.

We can parametrize the caps by G_0 and G_ℓ where, for each $t \in [0, \ell]$, we define $G_t : \mathcal{R} \to \mathbb{R}^n$ with

$$G_t(\vec{z}) = \alpha(t) + M(t) \begin{pmatrix} 0 \\ \vec{z} \end{pmatrix}.$$

Now G_0 induces an orientation that is opposite the boundary orientation on $\partial \mathcal{W}$ while G_t gives a compatible orientation. Hence, (11) becomes

$$\mathrm{Vol}_n(\mathcal{W}) = -\int_{I \times \mathbb{S}^{n-2}} (\alpha + F)^* \nu - \int_{\mathcal{R}} G_0^* \nu + \int_{\mathcal{R}} G_\ell^* \nu. \tag{12}$$

We calculate the integrals on the caps first. By the same reasoning as in Proposition 3.4, for each $t \in [0, \ell]$,

$$G_t^* \nu = \frac{1}{n} \det \big(G_t, \, [d(G_t)] \big) \, dz^1 \wedge dz^2 \wedge \cdots \wedge dz^{n-1}.$$

Now

$$\det\big(G_t,\,[d(G_t)]\big) = \det\left(\alpha(t) + M(t)\begin{pmatrix}0\\ z^1\\ z^2\\ \vdots\\ z^{n-1}\end{pmatrix}, M(t)\begin{pmatrix}0 & 0 & \cdots & 0\\ 1 & 0 & \cdots & 0\\ 0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 1\end{pmatrix}\right)$$
$$= \det(M(t))\det\left(M(t)^{-1}\alpha(t) + \begin{pmatrix}0\\ z^1\\ z^2\\ \vdots\\ z^{n-1}\end{pmatrix}, \begin{pmatrix}0 & 0 & \cdots & 0\\ 1 & 0 & \cdots & 0\\ 0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 1\end{pmatrix}\right)$$
$$= \det\left(M(t)^{-1}\alpha(t), \begin{pmatrix}0 & 0 & \cdots & 0\\ 1 & 0 & \cdots & 0\\ 0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 1\end{pmatrix}\right) = \alpha(t)\cdot\alpha'(t),$$

where the last equality holds by Lemma 4.1. Consequently,

$$\int\limits_{\mathcal{R}} G_\ell^*\nu - \int\limits_{\mathcal{R}} G_0^*\nu = \frac{1}{n}\operatorname{Vol}_{n-1}(\mathcal{R})(\alpha(\ell)\cdot\alpha'(\ell) - \alpha(0)\cdot\alpha'(0)). \tag{13}$$

Now we must calculate $\int\limits_{I\times\mathbb{S}^{n-2}}(\alpha+F)^*\nu$. Applying Proposition 3.4, over a coordinate patch of $\mathbb{S}^{n-2}$ with coordinate system $(x^1, x^2, \ldots, x^{n-2})$, we have

$$(\alpha+F)^*\nu = \frac{1}{n}\det\big(\alpha(t)+F(\vec{x}),\,\alpha'(t)+F_t(t,\vec{x}),\,M(t)[d\bar{H}]\big)\,dt\wedge dx^1\wedge\cdots\wedge dx^{n-2}$$

where here $F_t = \partial F/\partial t$. This can be broken down by multilinearity of the determinant as follows:

$$\begin{aligned}(\alpha+F)^*\nu = \frac{1}{n}\Big[&\det\big(\alpha(t),\,\alpha'(t),\,M(t)[d\bar{H}]\big)\\ &+\det\big(F(\vec{x}),\,F_t(t,\vec{x}),\,M(t)[d\bar{H}]\big)\\ &+\det\big(F(\vec{x}),\,\alpha'(t),\,M(t)[d\bar{H}]\big)\\ &+\det\big(\alpha(t),\,F_t(t,\vec{x}),\,M(t)[d\bar{H}]\big)\Big]\,dt\wedge dx^1\wedge\cdots\wedge dx^{n-2}.\end{aligned} \tag{14}$$

We now consider the integration over $[0,\ell]\times\mathbb{S}^{n-2}$ of the four forms in (14).

For the first determinant in (14),

$$\begin{aligned}\det\left(\alpha(t),\, \alpha'(t),\, M(t)[d\bar{H}]\right) &= \det\left(\alpha(t),\, M(t)\vec{e}_1,\, M(t)[d\bar{H}]\right)\\ &= \det(M(t))\det\left(M(t)^{-1}\alpha(t),\, \vec{e}_1,\, [d\bar{H}]\right)\\ &= -\det\left(\vec{e}_1,\, M(t)^{-1}\alpha(t),\, [d\bar{H}]\right).\end{aligned}$$

Doing Laplace expansion down the first column, we obtain an integral of the form in Proposition 3.7, with a vector $\vec{a}$ that depends on t. Hence, by Proposition 3.7,

$$\begin{aligned}&\int_{\mathbb{S}^{n-2}} \int_{t=0}^{\ell} \det\left(\alpha(t),\, \alpha'(t),\, M(t)[d\bar{H}]\right) dt \wedge dx^1 \wedge \cdots dx^{n-2}\\ &\qquad = (-1)^{n-2} \int_{t=0}^{\ell} \int_{\mathbb{S}^{n-2}} \det\left(\alpha(t),\, \alpha'(t),\, M(t)[d\bar{H}]\right) dx^1 \wedge \cdots dx^{n-2} \wedge dt = 0.\end{aligned}$$

For the second determinant in (14),

$$\begin{aligned}\det\left(F(t,\vec{x}),\, F_t(t,\vec{x}),\, M(t)[d\bar{H}]\right) &= \det\left(M(t)\bar{H}(\vec{x}),\, M(t)A(t)\bar{H}(\vec{x}),\, M(t)[d\bar{H}]\right)\\ &= \det(M(t))\det\left(\bar{H}(\vec{x}),\, A(t)\bar{H}(\vec{x}),\, [d\bar{H}]\right)\\ &= \det\left(\bar{H}(\vec{x}),\, A(t)\bar{H}(\vec{x}),\, [d\bar{H}]\right).\end{aligned}$$

Performing a Laplace expansion of the determinant using the first two columns of this last determinant produces terms resembling the forms described in Proposition 3.8. Since the centroid of $\mathcal{R}$ is assumed to be at the origin, then for all t, integrating all these terms over $\mathbb{S}^{n-2}$ give 0.

For the third determinant in (14) we have

$$\begin{aligned}\det\left(F(t,\vec{x}),\, \alpha'(t),\, M(t)[d\bar{H}]\right) &= \det\left(M(t)\bar{H}(\vec{x}),\, M(t)\vec{e}_1,\, M(t)[d\bar{H}]\right)\\ &= \det(M(t))\det\left(\bar{H}(\vec{x}),\, \vec{e}_1,\, [d\bar{H}]\right)\\ &= -\det\left(\vec{e}_1,\, \bar{H}(\vec{x}),\, [d\bar{H}]\right)\\ &= -\det\left(H(\vec{x}),\, [dH]\right)\end{aligned}$$

where the last equality follows by Laplace expansion of the determinant on the first row $\begin{pmatrix}1 & 0 & \cdots & 0\end{pmatrix}$. By Proposition 3.4,

$$\begin{aligned}&\int_{\mathbb{S}^{n-2}} \int_{t=0}^{\ell} \det\left(F(\vec{x}),\, \alpha'(t),\, [d(F)]\right)\, dt \wedge dx^1 \wedge \cdots \wedge dx^{n-2}\\ &\qquad = -\ell \int_{\mathbb{S}^{n-2}} \det\left(H(\vec{x}),\, [d(H)]\right)\, dx^1 \wedge \cdots \wedge dx^{n-2}\\ &\qquad = -(n-1)\ell\,\mathrm{Vol}_{n-1}(\mathcal{R}).\end{aligned}$$

As with previous determinants, the fourth determinant becomes

$$\det\left(\alpha(t),\, F_t(t,\vec{x}),\, M(t)[d\bar{H}]\right) = \det\left(M(t)^{-1}\alpha(t),\, A(t)\bar{H}(\vec{x}), [d\bar{H}]\right).$$

Since there are zeros in the first row of $\bar{H}$ and $d(\bar{H})$, and because $M^{-1} = M^\top$, another Laplace expansion gives

$$\begin{aligned}
&\det\left(M^{-1}\alpha,\, A\bar{H},\, [d\bar{H}]\right) \\
&\qquad = \alpha_i M_1^i \sum_{j=2}^{n} (-1)^j A_q^j \bar{H}^q \left|d_j \bar{H}\right| - \alpha_i \sum_{j=2}^{n} (-1)^j M_j^i A_q^1 \bar{H}^q \left|d_j \bar{H}\right| \\
&\qquad = \sum_{j=2}^{n} \alpha_i \bar{H}^q \left|d_j \bar{H}\right| (-1)^j \left[M_1^i A_q^j - M_j^i A_q^1\right],
\end{aligned}$$

where we use the Einstein summation convention over the repeated indices appearing in superscript and subscript, namely i and q. By Proposition 3.5, after integration on $\mathbb{S}^{n-2}$, all terms will reduce to 0 except those for which $q = j$, which will give the volume of the cross-section, $\mathrm{Vol}_{n-1}(\mathcal{R})$. Set $I = [0, \ell]$. So,

$$\begin{aligned}
&\int_{\mathbb{S}^{n-2}} \int_I \det\left(M^{-1}\alpha,\, A\bar{H},\, [d(\bar{H})]\right)\, dt \wedge dx^1 \wedge \cdots \wedge dx^{n-2} \\
&\qquad = \int_{\mathbb{S}^{n-2}} \int_I \sum_{j=2}^{n} \alpha_i \bar{H}^q \left|d_j \bar{H}\right| (-1)^j \left[M_1^i A_q^j - M_j^i A_q^1\right]\, dt \wedge dx^1 \wedge \cdots \wedge dx^{n-2} \\
&\qquad = \int_{\mathbb{S}^{n-2}} \int_I \sum_{j=2}^{n} \alpha_i \bar{H}^j \left|d_j \bar{H}\right| (-1)^j \left[M_1^i A_j^j - M_j^i A_j^1\right]\, dt \wedge dx^1 \wedge \cdots \wedge dx^{n-2} \\
&\qquad = \int_{\mathbb{S}^{n-2}} \int_I \sum_{j=2}^{n} \alpha_i H^{j-1} \left|d_j \bar{H}\right| (-1)^j M_j^i A_1^j\, dt \wedge dx^1 \wedge \cdots \wedge dx^{n-2} \\
&\qquad = \int_I \alpha_i \sum_{j=2}^{n} M_j^i A_1^j\, dt \int_{\mathbb{S}^{n-2}} (-1)^j H^{j-1} \left|d_{j-1} H\right|\, dx^1 \wedge \cdots \wedge dx^{n-2}.
\end{aligned}$$

But $\alpha'(t) = M(t)\vec{e}_1$ so $\alpha''(t) = M'(t)\vec{e}_1 = M(t)A(t)\vec{e}_1$. Hence $M_j^i A_1^j$ are the components of the covariant vector $\alpha''(t)^\top$. Note that since $A(t)$ is an antisymmetric matrix, $A_1^1 = 0$. Thus

$$\alpha_i \sum_{j=2}^{n} M_j^i A_1^j = \alpha(t) \cdot \alpha''(t).$$

Hence, we get

$$\begin{aligned}\int\limits_{\mathbb{S}^{n-2}} \int\limits_I \det\Big(M^{-1}\alpha,\, A\bar{H},\, [d(\bar{H})]\Big)\, dt \wedge dx^1 \wedge \cdots \wedge dx^{n-2}& \\ &= \operatorname{Vol}_{n-1}(\mathcal{R}) \int_I \alpha_i \alpha_i''\, dt \\ &= \operatorname{Vol}_{n-1}(\mathcal{R}) \left(\alpha(t)\cdot\alpha'(t)\Big|_0^\ell - \int_I \alpha'\cdot\alpha'\, dt \right) \\ &= \operatorname{Vol}_{n-1}(\mathcal{R}) \left(\alpha(t)\cdot\alpha'(t)\Big|_0^\ell - \ell \right).\end{aligned}$$

Now putting into (12) the integrals of the four determinants in (14) and the integrals for the caps (13), we get

$$\begin{aligned} n\cdot\operatorname{Vol}_n(\mathcal{W}) = (n-1)\operatorname{Vol}_{n-1}(\mathcal{R})\ell - \operatorname{Vol}_{n-1}(\mathcal{R}) \left(\alpha(t)\cdot\alpha'(t)\Big|_0^\ell - \ell \right) & \\ + \operatorname{Vol}_{n-1}(\mathcal{R}) \left(\alpha(t)\cdot\alpha'(t)\Big|_0^\ell \right).& \end{aligned} \tag{15}$$

Hence

$$\operatorname{Vol}_n(\mathcal{W}) = \operatorname{Vol}_{n-1}(\mathcal{R})\ell,$$

which establishes the main theorem when the guiding curve of the generalized tube is not closed.

Case 2: If C is a closed curve, then in (12) we do not have integrals for the caps. Then (15) becomes

$$n\cdot\operatorname{Vol}_n(\mathcal{W}) = (n-1)\operatorname{Vol}_{n-1}(\mathcal{R})\ell - \operatorname{Vol}_{n-1}(\mathcal{R}) \left(\alpha(t)\cdot\alpha'(t)\Big|_0^\ell - \ell \right)$$

and since $\alpha(0)\cdot\alpha'(0) = \alpha(\ell)\cdot\alpha'(\ell)$, the result of the main theorem follows for this case as well. □

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