W11 - Homework

Conditional distribution

01

Conditional density from joint density

Suppose that X and Y have joint probability density given by:

$$f_{X,Y}(x,y) \quad = \quad \begin{cases} \dfrac{12}{5}x(2-x-y) & x,y \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

(a) Compute $f_{X|Y}(x|y)$, for $y \in [0,1]$.

(b) Compute $P[\, X > 1/2 \mid Y = y \,]$.

02

From conditional to joint, and back again

Suppose we have the following data about random variables X and Y:

$$f_X(x) = \begin{cases} 3x^2 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_{Y|X}(y|x) = \begin{cases} 2y/x^2 & 0 \leq y \leq x \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the joint distribution $f_{X,Y}(x,y)$.

(b) Find $f_{X|Y}(x|y)$.

Conditional expectation

03

Conditional distribution and expectation from joint PMF

Suppose that X and Y have the following joint PMF:

$$P_{X,Y}(k,\ell) \quad = \quad \begin{cases} c & k = 1,2,3,4; \quad \ell = 1,\ldots,k \\ 0 & \text{otherwise} \end{cases}$$

Notice that the possibilities for ℓ depend on the choice of k.

First, show that $c = 1/10$ must be true. Then compute:

(a) P_X (b) $P_{Y|X}$ (c) $E[\, Y \mid X = 4 \,]$ (d) $E[\, Y \mid X \,]$

04

Conditional distribution and expectation from joint PDF

Suppose that X and Y have the following joint PDF:

$$f_{X,Y}(x,y) \quad = \quad \begin{cases} cxy & 0 < y < 1,\ 0 < x < y \\ 0 & \text{otherwise} \end{cases}$$

Notice that the range of possibilities for x depends on the choice of y.

First, show that $c = 8$ must be true. Then compute:

(a) f_X (b) $f_{Y|X}$ (c) $E[\,Y \mid X = 0.5\,]$ (d) $E[\,Y \mid X\,]$

05

"Plug In" Expectation Identity

Suppose $h(x, y)$ is a function, and X and Y are two random variables.

Verify this formula in the *continuous case*, using the definitions:

$$E[\,h(X,Y) \mid Y = y\,] \quad = \quad E[\,h(X,y) \mid Y = y\,]$$

Using that formula, prove this formula:

$$E[\,a(Y)b(X) \mid Y\,] = a(Y)\,E[\,b(X) \mid Y\,]$$

for two functions $a(y)$ and $b(x)$ and random variables X and Y. Notice that here the expectations are viewed as random variables.

Hint for second question: Both sides are functions of Y. Write these functions as $g_1(Y)$ and $g_2(Y)$ and check equality of the functions.

06

Iterated Expectation Identity

Prove the following identity using Iterated Expectation along with the previous exercise:

$$E[\,XY\,] = E[\,Y\,E[X \mid Y]\,]$$

Note: The solution is short once you find it – please clearly identify your choices for $a(y)$ and $b(x)$ functions.

07

★ How many customers buy a cake?

Let N count the number of customers that visit a bakery on a random day, and assume $N \sim \text{Pois}(\lambda)$.

Let X count the number of customers that make a purchase. Each customer entering the bakery smells the cakes, and this produces a probability p of buying a cake for that customer. The customers are independent.

Find $\text{Cov}[N, X]$. Are N and X positively or negatively correlated?

Hint: Compute $P_{X|N}(x|n)$, and use this to compute $E[\,X \mid N\,]$ in terms of N. Now deduce $E[X]$ using Iterated Expectation. Finally, compute $E[NX]$ using the Iterated Expectation Identity from the previous exercise. Now put everything together to find $\text{Cov}[N, X]$.