W10 - Homework

Functions of two variables, covariance, correlation

01

Covariance and correlation

The joint PMF of X and Y is given by the table:

$Y \downarrow \ X \rightarrow$	**0**	**1**	**2**	**3**
1	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{1}{15}$
2	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{10}$
3	$\frac{1}{30}$	$\frac{1}{30}$	0	$\frac{1}{10}$

Compute:

(a) $E[X+Y]$ (b) $E[\,(X-Y)^2\,]$ (c) $\mathrm{Cov}[X,Y]$ (d) $\rho[X,Y]$

02

Covariance etc. from independent densities

Suppose X and Y are independent variables with the following densities:

$$f_X(x) = \begin{cases} \frac{1}{3}e^{-x/3} & x > 0 \\ 0 & \text{otherwise} \end{cases} \qquad f_Y(y) = \begin{cases} \frac{1}{8}e^{-y/8} & y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Compute:

(a) $P[X > Y]$ (b) $E[XY]$ (c) $\mathrm{Cov}[X,Y]$ (d) $\rho[X,Y]$

03

Plumber completion time

A plumber is coming to fix the sink. He will arrive between 2:00 and 4:00 with uniform distribution in that range.

Sink fixes take an average of 45 minutes with completion times following an exponential distribution.

When do you expect the plumber to finish the job?

What is the variance for the finish time?

04

Correlation between overlapping coin flip sequences

Suppose a coin is flipped 30 times.

Let X count the number of heads among the first 20 flips, and Y count the heads in the last 20.

Find $\rho[X, Y]$.

Hint: Partition the flips into three groups of 10. Create *three* variables, counting heads, and express X and Y using these. What is the variance of a binomial distribution?

05

Variance puzzle: indicators

Suppose A and B are events satisfying:

$$P[A] = 0.5, \qquad P[B] = 0.2, \qquad P[AB] = 0.1$$

Let X count the number of these events that occurs. (So the possible values are $X = 0, 1, 2$.)

Find $\mathrm{Var}[X]$.

Hint: Try setting $X = X_A + X_B$.

06

★ When $\rho[X, Y] = 1$

Suppose $\rho[X, Y] = 1$ for two random variables X and Y.

Show that $Y = aX + b$, where $a = \sigma_Y / \sigma_X$.

Find the formula for b.

Hint: Study the proof that $-1 \leq \rho[X] \leq 1$, and think about $E[(\tilde{X} - \tilde{Y})^2]$.

(Note: A similar result and argument holds for $\rho[X, Y] = -1$.)

07

Further practice: Covariance etc. from joint density

Suppose X and Y are random variables with the following joint density:

$$f_{X,Y}(x, y) = \begin{cases} \frac{3}{2}\left(x^2 + y^2\right) & x, y \in [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

Compute:

(a) $E[X]$ (b) $E[Y]$ (c) $E[XY]$ (d) $\mathrm{Var}[X]$

(e) $\mathrm{Var}[Y]$ (f) $\mathrm{Cov}[X, Y]$ (g) $\rho[X, Y]$ (h) Are X and Y independent?

(It is worth thinking through which of these can be computed in multiple ways.)