W10 - Examples

Functions on two random variables

$E[\,X^2 + Y\,]$ from joint PMF chart

Suppose the joint PMF of X and Y is given by this chart:

$Y \downarrow \;\; X \rightarrow$	1	2
-1	0.2	0.2
0	0.35	0.1
1	0.05	0.1

Define $W = X^2 + Y$. Find the expectation $E[W]$.

Solution
First compute the values of W for each pair (X, Y) in the chart:

$Y \downarrow \;\; X \rightarrow$	1	2
-1	0	3
0	1	4
1	2	5

Now take the sum, weighted by probabilities:

$$\begin{array}{c} 0 \cdot (0.2) + 3 \cdot (0.2) + 1 \cdot (0.35) \\ +4 \cdot (0.1) + 2 \cdot (0.05) + 5 \cdot (0.1) \end{array} \quad \gg\gg \quad 1.95 \;=\; E[W]$$

$E[Y]$ two ways, and $E[XY]$, from joint density

Suppose X and Y are random variables with the following joint density:

$$f_{X,Y}(x,y) = \begin{cases} \frac{3}{16}xy^2 & x, y \in [0,2] \\ 0 & \text{otherwise} \end{cases}$$

(a) Compute $E[Y]$ using two methods.

(b) Compute $E[XY]$.

Solution
(a) <u>Method One:</u> via marginal PDF $f_Y(y)$:

$$f_Y(y) \quad = \quad \int_0^2 \frac{3}{16}xy^2\,dx \quad \gg\gg \quad \begin{cases} \frac{3}{8}y^2 & y \in [0,2 \\ 0 & \text{otherwise} \end{cases}$$

Then expectation:

$$E[Y] \;=\; \int_0^2 y\,f_Y(y)\,dy \quad \gg\gg \quad \int_0^2 \frac{3}{8}y^3\,dy \quad \gg\gg \quad 3/2$$

(a) <u>Method Two:</u> directly, via two-variable formula:

$$E[Y] \;=\; \int_0^2 \int_0^2 y \cdot \frac{3}{16}xy^2\,dy\,dx \gg\gg \quad \int_0^2 \frac{3}{4}x\,dx \quad \gg\gg \quad 3/2$$

(b) Directly, via two-variable formula:

$$\begin{aligned} E[XY] &= \int_0^2 \int_0^2 xy \cdot \frac{3}{16} xy^2 \, dy \, dx \\ &\gg\gg \int_0^2 \frac{3}{4} x^2 \, dx \quad \gg\gg \quad 2 \end{aligned}$$

Covariance from PMF chart

Suppose the joint PMF of X and Y is given by this chart:

$Y \downarrow \;\; X \rightarrow$	1	2
-1	0.2	0.2
0	0.35	0.1
1	0.05	0.1

Find $\mathrm{Cov}[X, Y]$.

Solution

We need $E[X]$ and $E[Y]$ and $E[XY]$.

$$\begin{aligned} E[X] &= 1 \cdot (0.2 + 0.35 + 0.05) + 2 \cdot (0.2 + 0.1 + 0.1) \quad \gg\gg \quad 1.4 \\ E[Y] &= -1 \cdot (0.2 + 0.2) + 0 \cdot (0.35 + 0.1) + 1 \cdot (0.05 + 0.1) \\ &\gg\gg \quad -0.25 \\ E[XY] &= -1 \cdot (0.2) - 2 \cdot (0.2) + 0 + 1 \cdot (0.05) + 2 \cdot (0.1) \quad \gg\gg \quad -0.35 \end{aligned}$$

Therefore:

$$\begin{aligned} \mathrm{Cov}[X, Y] &= E[XY] - E[X]E[Y] \\ &\gg\gg \quad -0.35 - (1.4)(-0.25) \quad \gg\gg \quad 0 \end{aligned}$$

Variance of sum of indicators

An urn contains 3 red balls and 2 yellow balls.

Suppose 2 balls are drawn without replacement, and X counts the number of red balls drawn.

Find $\mathrm{Var}(X)$.

Solution

Let X_1 indicate (one or zero) whether the first ball is red, and X_2 indicate whether the second ball is red, so $X = X_1 + X_2$.

Then $X_1 X_2$ indicates whether both drawn balls are red; so it is Bernoulli with success probability $\frac{3}{5} \frac{2}{4} = \frac{3}{10}$. Therefore $E[X_1 X_2] = \frac{3}{10}$.

We also have $E[X_1] = E[X_2] = \frac{3}{5}$.

The variance sum rule gives:

$$\begin{aligned} \mathrm{Var}[X] &= \mathrm{Var}[X_1] + \mathrm{Var}[X_2] + 2\mathrm{Cov}[X_1, X_2] \\ &\gg\gg \quad E[X_1^2] - E[X_1]^2 + E[X_2^2] - E[X_2]^2 + 2(E[X_1 X_2] - E[X_1]E[X_2]) \\ &\gg\gg \quad \frac{3}{5} - \left(\frac{3}{5}\right)^2 + \frac{3}{5} - \left(\frac{3}{5}\right)^2 + 2\left(\frac{3}{10} - \frac{3}{5} \cdot \frac{3}{5}\right) \quad \gg\gg \quad \frac{9}{25} \end{aligned}$$