W08 - Examples

Functions on two random variables

PMF of XY^2 from chart

Suppose the joint PMF of X and Y is given by this chart:

$Y \downarrow \;\; X \rightarrow$	1	2
-1	0.2	0.2
0	0.35	0.1
1	0.05	0.1

Define $W = XY^2$.

- (a) Find the PMF $P_W(w)$.
- (b) Find the expectation $E[W]$.

Max and Min from joint PDF

Suppose the joint PDF of X and Y is given by:

$$f_{X,Y}(x,y) \quad = \quad \begin{cases} \frac{3}{2}(x^2+y^2) & x,\, y \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Find the PDFs:

- (a) $W = \text{Max}(X,Y)$
- (b) $W = \text{Min}(X,Y)$

Solution

(a)

1. Compute CDF of W.
   - Convert to event form:

$$F_W(w) \quad = \quad P\big[\text{Max}(X,Y) \leq w\big]$$

   - Interpret:

$$\gg\gg \quad P\big[X \leq w \text{ and } Y \leq w\big]$$

   - Integrate PDF over the region, assuming $w \in [0,1]$:

$$\int_{-\infty}^{w} \int_{-\infty}^{w} f_{X,Y}(x,y)\,dx\,dy$$

   - Insert PDF formula:

$$\int_0^w \int_0^w \tfrac{3}{2}(x^2+y^2)\,dx\,dy \quad \gg\gg \quad w^4$$

2. Differentiate to find $f_W(w)$.

- $f_W = \frac{d}{dw} F_W(w)$:

$$f_W(w) \quad = \quad \begin{cases} 4w^3 & w \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

(b)

1. Compute CDF of W.
   - Convert to event form:

$$F_W(w) \quad = \quad P\big[\text{Min}(X,Y) \leq w\big]$$

   - Consider complement event to interpret:

$$\gg\gg \quad 1 - P\big[\text{Min}(X,Y) > w\big] \quad \gg\gg \quad 1 - P[X > w \text{ and } Y > w]$$

   - Integrate PDF over the region:

$$P[X > w \text{ and } Y > w] \quad \gg\gg \quad \int_w^1 \int_w^1 \tfrac{3}{2}(x^2 + y^2)\, dx\, dy$$

   - Compute integral:

$$\gg\gg \quad w^4 - w^3 - w + 1$$

   - Therefore:

$$F_W(w) \quad = \quad w + w^3 - w^4$$

2. Differentiate to find $f_W(w)$.
   - $f_W = \frac{d}{dw} F_W(w)$:

$$f_W(w) \quad = \quad \begin{cases} 1 + 3w^2 - 4w^3 & w \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

PDF of a quotient

Suppose the joint PDF of X and Y is given by:

$$f_{X,Y}(x,y) = \begin{cases} \lambda\mu e^{-(\lambda x + \mu y)} & x,\, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Find the PDF of $W = Y/X$.

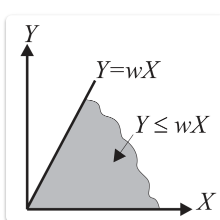
1. Find the CDF using logic.
   - Convert to event form:

$$F_W(w) \quad = \quad P[Y/X \leq w]$$

   - Re-express:

$$\gg\gg \quad P[Y \leq wX]$$

   - Diagram:



- Compute:

$$\begin{aligned} P[Y \leq wX] \quad &= \quad \int_0^\infty \int_0^{wx} f_{X,Y}(x,y)\,dy\,dx \\ \gg\gg \quad & \quad \int_0^\infty \lambda e^{-\lambda x} \int_0^{wx} \mu e^{-\mu y}\,dy\,dx \\ \gg\gg \quad & \quad \int_0^\infty \lambda e^{-\lambda x} \left(-e^{-\mu w x} + 1\right) dx \\ \gg\gg \quad & \quad 1 - \frac{\lambda}{\lambda + \mu w} \end{aligned}$$

2. ≡ Differentiate to find PDF.

- Compute $\frac{d}{dw} F_W(w)$:

$$f_W(w) = \begin{cases} \dfrac{\lambda\mu}{(\lambda+\mu w)^2} & w \geq 0 \\ \\ 0 & \text{otherwise} \end{cases}$$

Sums of random variables

Sum of parabolic random variables

Suppose X is an RV with PDF given by:

$$f_X(x) = \begin{cases} \frac{3}{4}(1-x^2) & x \in [-1,1] \\ 0 & \text{otherwise} \end{cases}$$

Let Y be an independent copy of X. So $f_Y = f_X$, but Y is independent of X.

Find the PDF of $X + Y$.

Solution

The graph of $f_X(w-x)$ matches the graph of $f_X(x)$ except (i) flipped in a vertical mirror, (ii) shifted by w to the left.

When $w \in [-2, 0]$, the integrand is nonzero only for $x \in [-1, w+1]$:

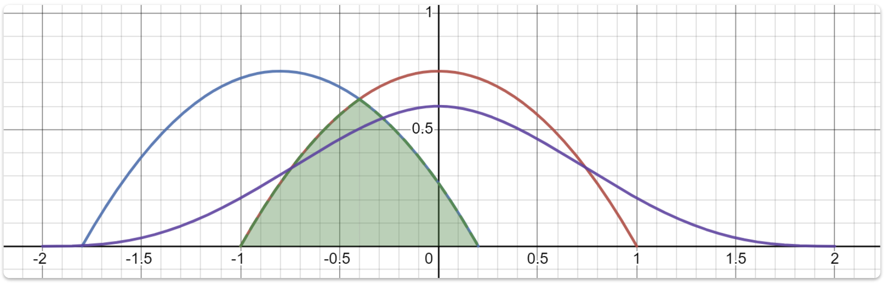
$$\begin{aligned} f_{X+Y}(w) \quad &= \quad \left(\frac{3}{4}\right)^2 \int_{-1}^{w+1} \left(1-(w-x)^2\right)\left(1-x^2\right) dx \\ &= \quad \frac{9}{16}\left(\frac{w^5}{30} - \frac{2w^3}{3} - \frac{4w^2}{3} + \frac{16}{15}\right) \end{aligned}$$

When $w \in [0, +2]$, the integrand is nonzero only for $x \in [w-1, +1]$:

$$\begin{aligned} f_{X+Y}(w) \quad &= \quad \left(\frac{3}{4}\right)^2 \int_{w-1}^{+1} \left(1-(w-x)^2\right)\left(1-x^2\right) dx \\ &= \quad \frac{9}{16}\left(-\frac{w^5}{30} + \frac{2w^3}{3} - \frac{4w^2}{3} + \frac{16}{15}\right) \end{aligned}$$

Final result is:

$$f_{X+Y}(w) \quad = \quad \begin{cases} \dfrac{9}{16}\left(\dfrac{w^5}{30} - \dfrac{2w^3}{3} - \dfrac{4w^2}{3} + \dfrac{16}{15}\right) & w \in [-2,0] \\ \\ \dfrac{9}{16}\left(-\dfrac{w^5}{30} + \dfrac{2w^3}{3} - \dfrac{4w^2}{3} + \dfrac{16}{15}\right) & w \in [0,2] \\ \\ 0 & \text{otherwise} \end{cases}$$



Discrete PMF formula for a sum

Prove the discrete formula for the PMF of a sum.
(Apply the general formula for the PMF of $g(X,Y)$.)

Vandermonde's identity from the binomial sum rule

Show that this "Vandermonde identity" holds for positive integers n, m, ℓ:

$$\sum_{j+k=\ell} \binom{n}{j}\binom{m}{k} = \binom{n+m}{\ell}$$

Hint: The binomial sum rule is:

$$\text{“Bin}(n,p) + \text{Bin}(m,p) \;\sim\; \text{Bin}(n+m,p)$$

Set $p = q = 1/2$. Compute the PMF of the left side using convolution. Compute the PMF of the right side directly. Set these PMFs equal.

Convolution practice

- Suppose X is an RV with density:

$$f_X = \begin{cases} 2x & x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

- Suppose Y is uniform on $[0,1]$.

Find the PDF of $X+Y$. Sketch the graph of this PDF.

Exp plus Exp equals Erlang

Let us verify this formula by direct calculation:

$$\text{“Exp}(\lambda) + \text{Exp}(\lambda) \quad = \quad \text{Erlang}(2,\lambda)\text{”}$$

Solution

Let X, $Y \sim \text{Exp}(\lambda)$ be independent RVs.

Therefore:

$$f_X = f_Y \quad = \quad \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Now compute the convolution:

$$f_{X+Y}(w) \quad = \quad \int_{-\infty}^{+\infty} f_X(w-x) f_Y(x)\,dx$$

$$\gg\gg \quad \int_0^w \lambda^2 e^{-\lambda(w-x)} e^{-\lambda x}\,dx \quad \gg\gg \quad \lambda^2 \int_0^w e^{-\lambda w}\,dx \quad \gg\gg \quad \lambda^2 w e^{-\lambda w}$$

This is the Erlang PDF:

$$f_X(t) = \frac{\lambda^\ell}{(\ell-1)!} t^{\ell-1} e^{-\lambda t}\bigg|_{\ell=2}$$

Erlang induction step

By direct computation with PDFs and convolution, derive the formula:

$$\text{“Exp}(\lambda) + \text{Erlang}(\ell, \lambda) \; = \; \text{Erlang}(\ell+1, \lambda)\text{”}$$

Combining normals

Suppose $X \sim \mathcal{N}(40, 16)$, $Y \sim \mathcal{N}(15, 9)$. Find the probability that $X \geq 2Y$.

Solution

Define $W = X - 2Y$. Using the formulas above, we see $W \sim \mathcal{N}(10, 52)$, or $W \sim \sqrt{52}Z + 10$ for a standard normal Z. Then:

$$P[X \geq 2Y] \quad \gg\gg \quad P[W \geq 0] \quad \gg\gg \quad P\left[Z \geq \tfrac{-10}{\sqrt{52}}\right]$$

$$\gg\gg \quad P[Z \leq 1.39] \quad \gg\gg \quad \approx 0.918$$