W05 - Homework

The symbol ★ indicates higher difficulty.

Poisson process

01

Poisson satisfies $F(+\infty) = 1$

Show that a Poisson variable $X \sim \text{Pois}(\lambda)$ satisfies the *total probability* rule for a CDF, namely that $\lim_{x\to\infty} F_X(x) = 1$.

02

Expectation of Poisson

Derive the formula $E[X] = \lambda$ for a Poisson variable $X \sim \text{Pois}(\lambda)$.

03

Application of Poisson: meteor shower

The UVA astronomy club is watching a meteor shower. Meteors appear at an average rate of 4 per hour.

- (a) Write a short explanation to justify the use of a Poisson distribution to model the appearance of meteors. Why should appearances be Poisson distributed?
- (b) What is the probability that the club sees more than 2 meteors in a single hour?
- (c) Suppose that over a four hour evening, 13 meteors were spotted. What is the probability that none of them happened in the first hour?

04

Silver dimes

Suppose 1 out of 350 dimes in circulation is made of silver. Consider a tub of dimes worth $40.

- (a) Find a formula for the exact probability that this collection contains at least 2 silver dimes. Can your calculator evaluate this formula?
- (b) Estimate the probability in question using a Poisson approximation.

(This topic for HW only, not for tests.)

05

Application of Poisson approximation of binomial

Let $X \sim \text{Bin}(10, \frac{1}{10})$ and consider the Poisson approximation to X.

- (a) Estimate the possible error of the approximation (for an arbitrary probability).
- (b) Compute the exact error of the approximation for the specific value $P[X \leq 1]$.

(This topic for HW only, not for tests.)

Function on a random variable

06

Constants in PDF from expectation

Suppose X has PDF given by:

$$f_X(x) = \begin{cases} a + bx^2 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Suppose $E[X] = \frac{7}{10}$. Find the only possible values for a and b. Then find $\text{Var}[X]$.

07

Variance: Direct integral formula

Suppose X has PDF given by:

$$f_X(x) = \begin{cases} 3e^{-3x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Find $\text{Var}[X]$ using the integral formula.

08

PDF of derived variable for $E[X]$ and $\text{Var}[X]$

Suppose the PDF of an RV is given by:

$$f_X(x) = \begin{cases} \frac{3}{4}x(2-x) & 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

- (a) Find $E[X]$ using the integral formula.
- (b) Find $f_{X^2}(x)$, the PDF of X^2 (by calculating the CDF first).
- (c) Find $E[X^2]$ using $f_{X^2}(x)$.
- (d) Find $\text{Var}[X]$ using results of (a) and (c).

Continuous wait times

09

Mean and variance of exponential

Show that $E[X] = \frac{1}{\lambda}$ and $\text{Var}[X] = \frac{1}{\lambda^2}$ for $X \sim \text{Exp}(\lambda)$.

10

★ Vehicle lifetimes

Suppose that vehicle lifetimes follow an exponential distribution with an expected lifetime of 10 years.

Suppose you have one car that is 5 years old, and one that is 15 years old.

What is the probability that the first car outlives the second?

11

★ Wait time for 5 calls - two methods

Consider the Poisson process of phone calls coming to a call center at an average rate of 1 call every 6 minutes.

Let us model the wait time for 5 calls to come in. You may use Desmos or similar to perform the integration numerically.

- (a) Method One: An arrival of '1-call' comes in at an average rate of $\lambda = 10$ calls per hour. So a Bundle of '5-calls' comes in at an average wait of $\lambda_B = 2$ Bundles per hour. Use an exponential variable with $\lambda_B = 2$ to determine the probability that the wait time for a Bundle (of 5 calls) is at most $1\,\text{hr}$.
- (b) Method Two: Use $\lambda = 10$ calls per hour with an Erlang distribution at $\ell = 5$ to determine the probability that the wait time for 5 calls is at most $1\,\text{hr}$.
- (c) Compare the results of (a) and (b). Can you explain why they agree or disagree? Which is correct??