W05 - Examples

Function on a random variable

33 - Expectation of function on RV given by chart

Suppose that $g : \mathbb{R} \to \mathbb{R}$ in such a way that $g : 1 \mapsto 4$ and $g : 2 \mapsto 1$ and $g : 3 \mapsto 87$.

$X:$	**1**	**2**	**3**
$P_X(k):$	$1/7$	$2/7$	$4/7$
$Y:$	4	1	87

Then:

$$E[X] = 1 \cdot \frac{1}{7} + 2 \cdot \frac{2}{7} + 3 \cdot \frac{4}{7} \quad \gg\gg \quad \frac{17}{7}$$

And:

$$E[Y] = 4 \cdot \frac{1}{7} + 1 \cdot \frac{2}{7} + 87 \cdot \frac{3}{7} \quad \gg\gg \quad \frac{267}{7}$$

Therefore:

$$E[5X + 2Y + 3] \quad \gg\gg \quad 5 \cdot \frac{17}{7} + 2 \cdot \frac{267}{7} + 3 \quad \gg\gg \quad \frac{640}{7}$$

34 - Variance of uniform random variable

The uniform random variable X on $[a, b]$ has distribution given by $P\big[X = [c, d]\big] = \frac{d-c}{b-a}$ for $[c, d] \subset [a, b]$.

- (a) Find $\mathrm{Var}[X]$.
- (b) Find $\mathrm{Var}[3X]$ using "squaring the scale factor."
- (c) Find $\mathrm{Var}[3X]$ directly.

Solution

(a)

1. ≡ Compute density.
    - The density for X is:

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{for } x \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$

2. ≡ Compute $E[X]$ and $E[X^2]$ directly using integral formulas.
    - Compute $E[X]$:

$$E[X] = \int_a^b \frac{x}{b-a}\, dx = \frac{b+a}{2}$$

- Now compute $E[X^2]$:

$$E[X^2] = \int_a^b \frac{x^2}{b-a}\,dx \quad \gg\gg \quad \frac{1}{3}(b^2+ba+a^2)$$

3. ≡ Find variance using short formula.
   - Plug in:

$$\text{Var}[X] = E[X^2] - E[X]^2$$

$$\gg\gg \quad \frac{1}{3}(b^2+ab+a^2) - \left(\frac{b+a}{2}\right)^2$$

$$\gg\gg \quad \frac{(b-a)^2}{12}$$

(b)

- "Squaring the scale factor" formula:

$$\text{Var}[aX+b] = a^2\text{Var}[X]$$

- Plugging in:

$$\text{Var}[3X] \quad \gg\gg \quad 9\text{Var}[X] \quad \gg\gg \quad \frac{9}{12}(b-a)^2$$

(c)

1. ≡ Density.
   - The variable $3X$ will have $1/3$ the density spread over the interval $[3a, 3b]$.
   - Density is then:

$$f_X(x) = \begin{cases} \frac{1}{3b-3a} & \text{on } [3a, 3b] \\ 0 & \text{otherwise} \end{cases}$$

2. ≡ Plug into prior variance formula.
   - Use $a \rightsquigarrow 3a$ and $b \rightsquigarrow 3b$.
   - Get variance:

$$\text{Var}[3X] = \frac{(3b-3a)^2}{12}$$

   - Simplify:

$$\gg\gg \quad \frac{(3(b-a))^2}{12} \quad \gg\gg \quad \frac{9}{12}(b-a)^2$$

35 - PDF of derived from CDF

Suppose that $F_X(x) = \frac{1}{1+e^{-x}}$.

- (a) Find the PDF of X.
- (b) Find the PDF of e^X.

Solution

(a)

- Formula:

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt \qquad \Longrightarrow \qquad f_X(x) = \frac{d}{dx} F_X(x)$$

- Plug in:

$$f_X(x) = \frac{d}{dx}\left(1 + e^{-x}\right)^{-1} \quad \gg\gg \quad -\left(1 + e^{-x}\right)^{-2} \cdot \left(-e^{-x}\right)$$

$$\gg\gg \quad \frac{e^{-x}}{\left(1 + e^{-x}\right)^2}$$

(b)

- By definition:

$$F_{e^X}(x) = P[e^X \leq x]$$

- Since e^X is increasing, we know:

$$e^X \leq a \qquad \Longleftrightarrow \qquad X \leq \ln a$$

- Therefore:

$$F_{e^X}(x) = F_X(\ln x)$$

$$\gg\gg \quad \frac{1}{1 + e^{-\ln x}} \quad \gg\gg \quad \frac{1}{1 + x^{-1}}$$

- Then using differentiation:

$$f_{e^X}(x) = \frac{d}{dx}\left(\frac{1}{1 + x^{-1}}\right)$$

$$\gg\gg \quad -(1 + x^{-1})^{-2} \cdot (-x^{-2}) \quad \gg\gg \quad \frac{1}{(x+1)^2}$$

Continuous wait time

37 - Earthquake wait time

Suppose the San Andreas fault produces major earthquakes modeled by a Poisson process, with an average of 1 major earthquake every 100 years.

- (a) What is the probability that there will *not* be a major earthquake in the next 20 years?
- (b) What is the probability that *three* earthquakes will strike within the next 20 years?

Solution

(a)

Since the average wait time is 100 years, we set $\lambda = 0.01$ earthquakes per year. Set

$X \sim \text{Exp}(0.01)$ and compute:

$$P[X > 20] = e^{-\lambda \cdot 20} \quad \gg\gg \quad e^{-0.01 \cdot 20} \quad \gg\gg \quad \approx 0.82$$

(b)
The same Poisson process has the same $\lambda = 0.01$ earthquakes per year. Set $X \sim \text{Erlang}(3, 0.01)$, so:

$$f_X(t) = \frac{\lambda^\ell}{(\ell - 1)!} t^{\ell-1} e^{-\lambda t}$$

$$\gg\gg \quad \frac{(0.01)^3}{(3-1)!} t^{3-1} e^{-0.01 \cdot t} \quad \gg\gg \quad \frac{10^{-6}}{2} t^2 e^{-0.01 \cdot t}$$

and compute:

$$P[X \leq 20] = \int_0^{20} f_X(x)\, dx$$

$$\gg\gg \quad \int_0^{20} \frac{10^{-6}}{2} t^2 e^{-0.01 \cdot t}\, dt \quad \gg\gg \quad \approx 0.00115$$