W03 - Examples

Repeated trials

19 - Multinomial: Soft drinks preferred

Folks coming to a party prefer Coke (55%), Pepsi (25%), or Dew (20%). If 20 people order drinks in sequence, what is the probability that exactly 12 have Coke and 5 have Pepsi and 3 have Dew?

Solution

The multinomial coefficient $\binom{20}{12, 5, 3}$ gives the number of ways to assign 20 people into bins according to preferences matching the given numbers, $C = 12$ and $P = 5$ and $D = 3$.

Each such assignment is one sequence of outcomes. All such sequences have probability $(0.55)^{12} \cdot (0.25)^5 \cdot (0.2)^3$.

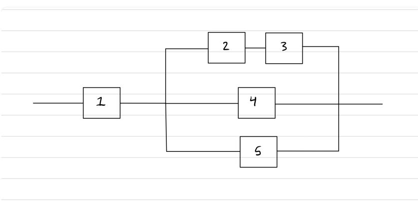
The answer is therefore:

$$\binom{20}{12, 5, 3} \cdot (0.55)^{12} \cdot (0.25)^5 \cdot (0.2)^3 \quad \gg\gg \quad \frac{20!}{12!\,5!\,3!} \cdot (0.55)^{12} \cdot (0.25)^5 \cdot (0.2)^3$$

Reliability

20 - Reliability: Series, parallel, series

Suppose a process has internal components arranged like this:



Write W_i for the event that component i succeeds, and W_i^c for the event that it fails.

The success probabilities for each component are given in the chart:

1	2	3	4	5
92%	89%	95%	86%	91%

Find the probability that the entire system succeeds.

Solution

1. Conjoin components 2 and 3 in series.
   - Compute:

   $$P[W_2 W_3] \quad \gg\gg \quad P[W_2] \cdot P[W_3] \quad \gg\gg \quad (0.89) \cdot (0.95) = 0.8455$$

   - Therefore:

   $$P\big[(W_2 W_3)^c\big] \quad \gg\gg \quad 1 - 0.846 \quad \gg\gg \quad 0.1545$$

2. Conjoin components (2-3) with 4 and 5 in parallel.
    - Compute for the complement (failure) first:

$$P\big[(W_2W_3 \cup W_4 \cup W_5)^c\big] \quad \gg\gg \quad P[(W_2W_3)^c] \cdot P[W_4^c] \cdot P[W_5^c]$$

$$\gg\gg \quad (0.1545)(0.14)(0.09) \quad \gg\gg \quad 0.0019467$$

    - Flip back to success:

$$P\big[W_2W_3 \cup W_4 \cup W_5\big] \quad \gg\gg \quad 1 - 0.0019467 \quad \gg\gg \quad 0.9980533$$

3. Conjoin components 1 with (2-3-4-5) in series.
    - Compute:

$$P\Big[W_1\big(W_2W_3 \cup W_4 \cup W_5\big)\Big] \quad \gg\gg \quad (0.92)(0.9980533)$$

$$\gg\gg \quad 0.918209036 \quad \approx 91.82\%$$

Discrete random variables

21 - PDF and CDF: Roll 2 dice

Roll two dice colored red and green. Let X_R record the number of dots showing on the red die, X_G the number on the green die, and let S be a random variable giving the total number of dots showing after the roll, namely $S = X_R + X_G$.

- Find the PMFs of X_R and of X_G and of S.
- Find the CDF of S.
- Find $P[S = 8]$.

Solution

1. Sample space.
    - Denote outcomes with ordered pairs of numbers (i, j), where i is the number showing on the red die and j is the number on the green one.
    - Require that $i, j \in \mathbb{N}$ are integers satisfying $1 \leq i,\ j \leq 6$.
    - Events are sets of distinct such pairs.
2. Create chart of outcomes.
    - Chart:

+	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

3. Definitions of X_R, X_G, and S.
    - We have $X_R(i, j) = i$ and $X_G(i, j) = j$.
    - Therefore $S(i, j) = i + j$.
4. Find PMF of X_R.
    - Use variable n for each possible value of X_R, so $n = 1,\ 2,\ \ldots,\ 6$.

- Find $P_{X_R}(n)$:

$$P_{X_R}(n) \quad \gg\gg \quad P[X_R = n]$$

$$\gg\gg \quad \frac{|\text{outcomes with } n \text{ on red}|}{\text{all outcomes}} \quad \gg\gg \quad \frac{6}{36} = \frac{1}{6}$$

- Therefore $P_{X_R}(n) = 1/6$ for every n.

5. Find PMF of X_G.
   - Same as for X_R:

$$P_{X_G}(n) = \frac{1}{6} \quad \text{for all } n$$

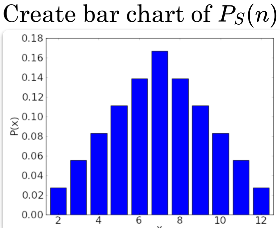
6. Find PMF of S.
   - Find $P_S(n)$:

$$P_S(n) \quad \gg\gg \quad P[S = n] \quad \gg\gg \quad \frac{|\text{outcomes with sum } n|}{\text{all outcomes}}$$

   - ⚠ Count outcomes along *diagonal lines* in the chart.
   - Create table of $P_S(n)$:

k	2	3	4	5	6	7	8	9	10	11	12
$p_S(k) = P(S = k)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

   - Create bar chart of $P_S(n)$:



   - Evaluate: $P[S = 8] \quad \gg\gg \quad 5/36$.

7. Find CDF of S.
   - CDF definition:

$$F_S(n) = P[S \leq n]$$

   - Apply definition: add new PMF value at each increment:

$$F_S(n) = \begin{cases} 0 & n < 1 \\ 1/36 & 1 \leq n < 2 \\ 3/36 & 2 \leq n < 3 \\ 6/36 & 3 \leq n < 4 \\ 10/36 & 4 \leq n < 5 \\ 15/36 & 5 \leq n < 6 \\ 21/36 & 6 \leq n < 7 \\ 26/36 & 7 \leq n < 8 \\ 30/36 & 8 \leq n < 9 \\ 33/36 & 9 \leq n < 10 \\ 35/36 & 10 \leq n < 11 \\ 36/36 & 11 \leq n \end{cases}$$

22 - PMF for total heads count; binomial expansion of 1

A fair coin is flipped n times.

Let X be the random variable that counts the total number of heads in each sequence.

The PMF of X is given by:

$$P_X(k) = \binom{n}{k} \left(\frac{1}{2}\right)^n$$

Since the total probability must add to 1, we know this formula must hold:

$$1 = \sum_{\text{possible } k} P_X(k)$$

$$\gg\gg \quad 1 = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{2}\right)^n$$

Is this equation really true?

There is another way to view this equation: it is the binomial expansion $(x+y)^n$ where $x = \frac{1}{2}$ and $y = \frac{1}{2}$:

$$\left(\frac{1}{2} + \frac{1}{2}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{2}\right)^n$$

23 - Life insurance payouts

A life insurance company has two clients, A and B, each with a policy that pays \$100,000 upon death. Consider events D_1 that the older client dies next year, and D_2 that the younger dies next year. Suppose $P[D_1] = 0.10$ and $P[D_2] = 0.05$.

Define a random variable X measuring the total money paid out next year in units of \$1,000. The possible values for X are 0, 100, 200. We calculate:

$$\begin{aligned} P[X=0] &\quad \gg\gg \quad P[D_1^c]P[D_2^c] = 0.95 \cdot 0.90 = 0.86 \\ P[X=100] &\quad \gg\gg \quad 0.05 \cdot 0.90 + 0.95 \cdot 0.10 = 0.14 \\ P[X=200] &\quad \gg\gg \quad 0.05 \cdot 0.10 = 0.005 \end{aligned}$$