Unit 03 - Essential problems

W12

Counting flip flops

A bag contains 50 marbles, 30 blue and 20 red. A sequence of zeros and ones is created by pulling the marbles out one at a time (without replacement) and writing a 1 if the marble drawn is blue and a zero if it is red.

How many pairs of adjacent digits in the sequence are expected to differ from each other?

Hint: Use a sum of 49 indicators.

Normal approximation - Eating hot dogs

Frank is a competitive hot dog eater. He eats $1\,\text{hd}$ in $15\,\text{sec}$ with $\sigma = 4\,\text{sec}$.

What is the probability that Frank manages to consume $64\,\text{hd}$ in $15\,\text{min}$ or less, in an upcoming competition? Use a normal approximation from the CLT to estimate this probability.

State the reason that the normal approximation is applicable.

Continuity correction: De Moivre-Laplace

A fair die is rolled 300 times.

Use a normal approximation to estimate the probability that exactly 100 outcomes are either 3 or 6.

Do this with and without the continuity correction.

Normal approximation - Grading many exams

An instructor has 50 exams to grade. The grading time for each exam follows a distribution with an average of 20 minutes and variance of 16 minutes. Assume the grading times per exam are independent.

Roughly what are the odds that after 450 minutes of grading, at least half the exams will be graded? Use a normal approximation to estimate the answer.

State the reason that the normal approximation is applicable.

W13

Normal approximation: Heads v. tails

Flip a coin 10,000 times. Let H measure the number of heads, and T measure the number of tails. Estimate the probability that H and T are within 100 of each other.

Hint: Write an inequality for the condition, then sub a formula for T in terms of H.

Deviation estimation - Exponential

Let $X \sim \text{Exp}(\lambda)$ with $\lambda = 0.5$.

(a) Compute the Markov bound on $P[X > 5]$.

(b) Compute the Chebyshev bound on $P[X > 5]$.

(c) Find the exact value of $P[X > 5]$ and compare with yours answers in (a) and (b).

Deviation estimation - Factory production

Suppose a factory produces an average of 50 items per week.

(a) How likely is it that more than 75 items are produced this week? (Find an upper bound.)

(b) Suppose the variance is known to be 25 items. Now what can you say about (a)? (Hint: Monotonicity.)

(c) What do you know about the probability that the number of items produced differs from the average by at most 10?

Testing paperclips - Likelihood of error

A factory assembly line machine is cutting paperclips to length before folding. Each paperclip is supposed to be $3\,\text{in}$ long. The length of paperclips is approximately normally distributed with standard deviation $0.2\,\text{in}$.

(a) Design a significance test with $\alpha = 0.02$ that is based on the average of 5 measurements (sample mean). What is the rejection region? What is the probability of Type I error?

(b) What is the probability of Type II error, given that the average paperclip length on the machine is actually $3.1\,\text{in}$?

W14

Significance test: Blue eyes

A redditor claims that 10% of people have blue eyes, but you think it is not that many. You work at the DMV for the summer, so you write down the eye color recorded on drivers' licenses of various people in the database.

(a) Suppose you record the eye color of 1000 people and let X be the number that are blue. If the rejection region is $\{X \leq 85\}$, what is the significance level of the test?

(b) Take again the experiment in (a). If you want a significance level of $\alpha = 0.01$, what should the rejection region be in your test?

(c) Suppose the fact is that 7% of people have blue eyes. How likely is it that your test in (b) rejects H_0?

Binary hypothesis test: Identifying Uranium

You are testing gram samples of pure Uranium to see if they are enriched. You have a Geiger counter that counts a number of gamma rays that come from nearby fission events in 1 second intervals after you press the count button.

If the sample is enriched, you expect a Poisson distribution N of gamma rays in the counter with an average of 20. If the sample is not enriched (the null hypothesis), the average count will be 10.

(a) Design an ML test to decide whether it is ordinary (H_0) or enriched (H_1). What is A_0? What are the probabilities of Type I, Type II, and Total error?

(b) After running the test many times, you have noticed that 70% of the samples are ordinary, while 30% are enriched. Now design an MAP test. What is A_0? What are the probabilities of Type I, Type II, and Total error?

(c) Missing a bit of enriched Uranium is obviously a major problem. The damage to your reputation and pocketbook of missing enriched Uranium is $100\times$ the damage caused by incorrectly labeling ordinary Uranium as enriched. Now design an MC test. What is A_0? What are the probabilities of Type I, Type II, and Total error?

(d) What is the expected cost of each application of the MC test, assuming the cost of a false alarm is \$10,000? What is this number for the MAP test?

W15

Estimates from joint PDF

Suppose X and Y have the following joint PDF:

$$f_{X,Y}(x,y) = \begin{cases} \frac{6(y-x)}{27} & 0 \le x \le y \le 3 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find $f_X(x)$ and the blind estimate $\hat{x}_B$.

(b) Compute $\hat{x}_G$, the MMSE estimate of X assuming the event $G = \{X < 3/2\}$.

(c) Find $f_Y(y)$ and the blind estimate $\hat{y}_B$.

(d) Compute $\hat{y}_H$, the MMSE estimate of Y assuming the event $H = \{Y > 3/2\}$.

MMSE exact estimator from joint PDF

Suppose X and Y have the following joint PDF:

$$f_{X,Y}(x,y) = \begin{cases} 2(y+x) & 0 \le x \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) What is $\hat{x}_M(y)$, the MMSE estimate of X given $Y = y$?

(b) What is $\hat{y}_M(x)$, the MMSE estimate of Y given $X = x$?

MMSE linear estimator from joint PMF

Suppose X and Y have the following joint PMF:

$Y \downarrow \ X \rightarrow$	-1	0	1
1	$\frac{1}{6}$	$\frac{1}{12}$	0
3	$\frac{1}{8}$	$\frac{1}{12}$	$\frac{1}{24}$
5	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{1}{8}$
7	0	$\frac{1}{12}$	$\frac{1}{6}$

(a) Find the minimal MSE linear estimator for X in terms of Y.

(b) What is the MMSE error for this linear estimator?

(c) Use (a) to estimate X given $Y = 1$ and $Y = 5$.