Unit 02 - Essential problems

W07

PMF calculations from a table

Suppose the joint PMF of X and Y has values given in this table:

$X\backslash Y$	**0**	**1**	**2**	**3**
1	0.10	0.15	0	0.05
2	0.20	0.05	0.05	0.20
3	0.05	0	x	0.05

- (a) Find x.
- (b) Find the marginal PMF of X.
- (c) Find the PMF of the random variable $Z = XY$.
- (d) Find $P[X = Y]$ and $P[X > Y]$.

Marginals from PDF

Suppose X and Y have joint PDF given by:

$$f_{X,Y}(x,y) = \begin{cases} 2e^{-(x+2y)} & \text{if } x, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

- (a) Find the marginal PDFs for X and Y.
- (b) Find $P[X > Y]$.

★ Factorizing the density

Consider two joint density functions for X and Y:

$$\begin{aligned} f_1(x,y) &= 6e^{-2x}e^{-3y}, \qquad x, y > 0, \\ f_2(x,y) &= 2yxe^{x^2}, \qquad x, y \in [0,1],\ x + y \in [0,1]. \end{aligned}$$

(Assume the densities are zero outside the given domain.)

Supposing f_1 is the joint density, are X and Y independent? Why or why not?
Supposing f_2 is the joint density, are X and Y independent? Why or why not?

★ Composite PDF from joint PDF

The joint density of random variables X and Y is given by:

$$f_{X,Y}(x,y) \quad = \quad \begin{cases} e^{-x-y} & x, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Compute the PDF of X/Y. (Hint: First find the CDF of X/Y.)

W08

PDF of min and max

Suppose $X \sim \text{Exp}(2)$ and $Y \sim \text{Exp}(3)$ and these variables are independent. Find:

- (a) The PDF of $W = \text{Max}(X, Y)$
- (b) The PDF of $W = \text{Min}(X, Y)$

PDF of sum from joint PDF

Suppose the joint PDF of X and Y is given by:

$$f_{X,Y} \quad = \quad \begin{cases} \frac{8}{81} xy & 0 \leq y \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

Find the PDF of $X + Y$.

★ Poisson plus Bernoulli

Suppose that:

- $X \sim \text{Pois}(\lambda)$
- $Y \sim \text{Ber}(p)$
- X and Y are independent

Find a formula for the PMF of $X + Y$.

Apply your formula with $\lambda = 2$ and $p = 0.3$ to find $P_{X+Y}(7)$.

Convolution for uniform distributions over intervals

Suppose that:

- $X \sim \text{Unif}[a, b]$
- $Y \sim \text{Unif}[c, d]$
- X and Y are independent

Find the PDF of $X + Y$.

(You may find it helpful to start by considering specific numbers for a, b, c, d.)

W10

Correlation between overlapping coin flip sequences

Suppose a coin is flipped 30 times.

Let X count the number of heads among the first 20 flips, and Y count the heads in the last 20.

Find $\rho[X, Y]$.

Hint: Partition the flips into three groups of 10. Create *three* variables, counting heads, and express X and Y using these. What is the variance of a binomial distribution?

Variance puzzle: indicators

Suppose A and B are events satisfying:

$$P[A] = 0.5, \qquad P[B] = 0.2, \qquad P[AB] = 0.1$$

Let X count the number of these events that occurs. (So the possible values are $X = 0, 1, 2$.)

Find $\mathrm{Var}[X]$.

Hint: Try setting $X = X_A + X_B$.

Further practice: Covariance etc. from joint density

Suppose X and Y are random variables with the following joint density:

$$f_{X,Y}(x,y) = \begin{cases} \frac{3}{2}\left(x^2 + y^2\right) & x, y \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Compute:

(a) $E[X]$ (b) $E[Y]$ (c) $E[XY]$ (d) $\mathrm{Var}[X]$

(e) $\mathrm{Var}[Y]$ (f) $\mathrm{Cov}[X, Y]$ (g) $\rho[X, Y]$ (h) Are X and Y independent?

(It is worth thinking through which of these can be computed in multiple ways.)

W11

Conditional density from joint density

Suppose that X and Y have joint probability density given by:

$$f_{X,Y}(x,y) \quad = \quad \begin{cases} \frac{12}{5} x(2 - x - y) & x, y \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

(a) Compute $f_{X|Y}(x|y)$, for $y \in [0,1]$.

(b) Compute $P[\, X > 1/2 \mid Y = y \,]$.

Conditional distribution and expectation from joint PDF

Suppose that X and Y have the following joint PDF:

$$f_{X,Y}(x,y) \;=\; \begin{cases} cxy & 0<y<1,\; 0<x<y \\ 0 & \text{otherwise} \end{cases}$$

Notice that the range of possibilities for x depends on the choice of y.

First, show that $c = 8$ must be true. Then compute:

(a) f_X (b) $f_{Y|X}$ (c) $E[\,Y \mid X = 0.5\,]$ (d) $E[\,Y \mid X\,]$

★ How many customers buy a cake?

Let N count the number of customers that visit a bakery on a random day, and assume $N \sim \text{Pois}(\lambda)$.

Let X count the number of customers that make a purchase. Each customer entering the bakery smells the cakes, and this produces a probability p of buying a cake for that customer. The customers are independent.

Find $\text{Cov}[N, X]$. Are N and X positively or negatively correlated?

Hint: Compute $P_{X|N}(x|n)$, and use this to compute $E[\,X \mid N\,]$ in terms of N. Now deduce $E[X]$ using Iterated Expectation. Finally, compute $E[NX]$ using the Expectation Multiplication Rule from the previous exercise. Now put everything together to find $\text{Cov}[N, X]$.