Unit 01 - Essential problems

W01

Venn diagrams - set rules and Kolmogorov additivity

Suppose we know three probabilities of events: $P[A] = 0.4$, $P[B] = 0.3$, and $P[A \cap B] = 0.1$.

Calculate: $P[A \cup B]$, $P[A^c]$, $P[B^c]$, $P[A \cap B^c]$, and $P[(A \cap B)^c]$.

Inclusion-exclusion reasoning

Your friend says: "according to my calculations, the probability of A is 0.5 and the probability of B is 0.7, but the probability of A and B both happening is only 0.001."

You tell your friend they don't understand probability. Why?

At least two heads from three flips

A coin is flipped three times.

What is the probability that at least two heads appear?

Conditioning - two dice, at least one is 5

Two dice are rolled, and at least one is a 5.

What is the probability that their sum is 10?

Multiplication - drawing two hearts

Two cards are drawn from a standard deck (without replacement).

(a) What is the probability that both are hearts?

(b) What is the probability that both are 4?

W02

Conditioning relation

Suppose you know $P[A \cap B] = 0.036$ and $P[A \mid B] = 0.18$ and $P[B \mid A] = 0.60$.

Calculate $P[A]$ and $P[B]$ and $P[A \cup B]$.

Bayes' Theorem - Inferring die from roll

A bag contains one 4-sided die, one 6-sided die, and one 12-sided die. You draw a random die from the bag, roll it, and get a 4.

What is the probability that you drew the 6-sided die?

Bayes' Theorem - DNA evidence

A crime is committed in a town of 100,000 citizens. After all 100,000 citizens' DNA is analyzed, your friend Jim is found to have a DNA match to evidence at the scene. A forensics expert says that the probability of a random person matching this evidence is 0.01%. How likely is it that Jim is guilty?

Bin of marbles

A bin contains 5 red marbles, 7 blue marbles, and 3 white marbles.

We draw a random marble. If it's red, we put it back, if it's not red, we keep it. We do this three times.

(a) What is the probability of getting red then white then blue?

(b) Suppose the last draw was blue. What is the probability that the first was red?

Counting outcomes - permutations and combinations

In a lottery, five distinct numbers are picked at random from $1, 2, 3, \ldots, 40$. How many possible outcomes are there:

(a) If we care about the order of numbers.

(b) If the order does not matter.

W03

Multinomial - Colored marbles in a line

How many ways are there to line up 10 colored marbles (2 red, 3 white, 5 blue), assuming you cannot distinguish marbles of the same color?

Independent trials - At least 45 good paper clips

For a paper clip production line, 90% of the paper clips come off good, and 10% come off broken.

You buy a box of 50 paper clips from this line. What is the probability that at least 45 of them are good?

Guessing on a test

Your odds of getting any given exam question right are 80%. The exam has 4 questions, and you need to answer 3 correctly to pass.

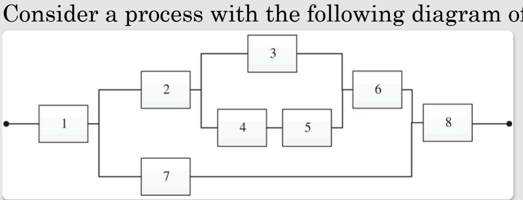
(a) What is the probability that you pass?

(b) After finishing the exam, you are 100% sure that you got the second question right. Now what are the odds that you pass?

W04

Reliability for complex process

Consider a process with the following diagram of components in series and parallel:



Use W_i to denote the event that component i succeeds.

Suppose the success probabilities per component are given by this chart:

1	2	3	4	5	6	7	8
80%	60%	40%	90%	80%	50%	70%	90%

What are the odds of success for the whole process?

Rolling until a six

A fair die is rolled until a six comes up.

What are the odds that it takes at least 10 rolls? Use a geometric random variable.

Prize on the Mall

A booth on the Mall is running a secret prize game, in which the 5^{th} passerby wearing a hat wins $1,000.

Passersby wear hats independently of each other and with probability 20%.

Let N be a random variable counting how many passersby pass by before a winner is found.

(a) What is the name of the distribution for N? What are the parameters?

(b) What is the probability that the 10^{th} passerby wins the prize?

(c) What is the probability that at least 7 passersby are needed before a winner is found?

Insurance expected payout

A car insurance analytics team estimates that the cost of repairs per accident is uniformly distributed between $100 and $1500. The manager wants to offer customers a policy that has a $500 deductible and covers all costs above the deductible.

How much is the expected payout per accident?

(Hint: Graph the PDF for the cost of repairs X; write a formula for the payout in terms of X using cases; then integrate.)

W05

Application of Poisson: meteor shower

The UVA astronomy club is watching a meteor shower. Meteors appear at an average rate of 4 per hour.

(a) Write a short explanation to justify the use of a Poisson distribution to model the appearance of meteors. Why should appearances be Poisson distributed?

(b) What is the probability that the club sees more than 2 meteors in a single hour?

(c) Suppose that over a four hour evening, 13 meteors were spotted. What is the probability that none of them happened in the first hour?

Constants in PDF from expectation

Suppose X has PDF given by:

$$f_X(x) = \begin{cases} a + bx^2 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Suppose $E[X] = \frac{7}{10}$. Find the only possible values for a and b. Then find $\text{Var}[X]$.

Variance: Direct integral formula

Suppose X has PDF given by:

$$f_X(x) = \begin{cases} 3e^{-3x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Find $\text{Var}[X]$ using the integral formula.

PDF of derived variable for $E[X]$ and $\text{Var}[X]$

Suppose the PDF of an RV is given by:

$$f_X(x) = \begin{cases} \frac{3}{4}x(2-x) & 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find $E[X]$ using the integral formula.

(b) Find $f_{X^2}(x)$, the PDF of X^2 (by calculating the CDF first).

(c) Find $E[X^2]$ using $f_{X^2}(x)$.

(d) Find $\text{Var}[X]$ using results of (a) and (c).

W06

Generalized normal - Table practice

Let X be generalized normal variable with $\mu = 3$ and $\sigma = 2$. Using a chart of Φ values, find:

(a) $P[2 < X < 6]$

(b) c such that $F_X(c) = 0.67$

(c) $E[X^2]$ (Hint: Use μ and σ to avoid integration.)

Normal distribution - cars passing toll booth

The number of cars passing a toll booth on Wednesdays has a normal distribution $\mathcal{N}(1200, 40000)$.

(a) What is the probability that on a randomly chosen Wednesday, more than 1,400 cars pass the toll booth?

(b) What is the probability that between 1,000 and 1,400 cars pass the toll booth on a random Wednesday?

(c) Suppose it is also known that at least 1200 cars passed the toll booth last Wednesday. What is the probability that at least 1300 cars passed the toll booth that day?