CALCULUS and PROBABILITY; a REVIEW

Calculus is an essential tool in introductory probability theory. The aspects discussed here represent the minimum knowledge of calculus needed for success. You will use them repeatedly and will want to have them in mind when doing homework and examinations.

Infinite Series

You will repeatedly encounter a small cast of infinite series in discrete probability problems. The most important series are the geometric and the exponential. They are given by;

Geometric

$$1 + x + x^2 + x^3 + \cdots = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x},$$

convergent for $|x| < 1$.

Exponential

$$1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x,$$

convergent for all x.

These series can be used (without going through Taylor's formula) for getting the series for other functions. For example, differentiating the geometric series term by term, we get the identity, valid for $|x| < 1$,

$$\sum_{k=0}^{\infty} kx^{k-1} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}.$$

We can differentiate again to deduce that for $|x| < 1$,

$$\sum_{k=0}^{\infty} k(k-1)x^{k-2} = \frac{d^2}{dx^2}\left(\frac{1}{1-x}\right) = \frac{2}{(1-x)^3}.$$

These ideas will be used repeatedly!

Another useful idea is substitution. Replacing x by $x^2/2$ in the exponential series, we learn that

$$\sum_{k=0}^{\infty} \frac{x^{2k}}{2^k k!} = \exp\left(x^2/2\right)$$

and, if we just replace x by $-x$, we learn that

$$\sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!} = e^{-x},$$

A final series, actually a polynomial, is the

Binomial

$$(1+x)^n = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} x^k = \sum_{k=0}^{n} \binom{n}{k} x^k,$$

where of course $\binom{n}{k}$ is the binomial coefficient.

This result is easily verified using Taylor's Theorem

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k.$$

Problems: 1. The reader is invited to use the Binomial Series to show that the terms in the series for $(1+\frac{x}{n})^n$ converge to the terms in the series for e^x, as $n \to \infty$; this illustrates that

$$\lim_{n\to\infty} (1+\frac{x}{n})^n = e^x.$$

2. Evaluate the series a. $\sum_0^{\infty} n\,(2/3)^n$, b. $\sum_{k=0}^{\infty} \frac{1}{k!}(-3)^k$ and c. $\sum_{j=0}^{20} \binom{20}{j}(-2)^j$.

Fundamental Theorem of Calculus

Part of the fundamental theorem of calculus says that if f is a sufficiently nice function, then

$$\frac{d}{dx}\left(\int_a^x f(x')dx'\right) = f(x)$$

i.e., the derivative of the integral is the integrand, *evaluated at* x.

We can thus go back and forth between

$$F(x) = \int_{-\infty}^{x} f(x')dx' \text{ and } f(x) = F'(x).$$

If we know, for example, that $F(x) = 0$, $x \le 0$, and $= 1 - e^{-x}$, $x > 0$, then evidently the function $f(x)$ is

$$
\begin{aligned}
f(x) &= \frac{d}{dx}F(x) \\
&= 0,\ x \le 0 \\
& \quad e^{-x},\ x > 0
\end{aligned}
$$

Problem: Suppose that

$$
\begin{aligned}
F(x) = \int_0^x f(y)\,dy &= 0,\ x \le 0 \\
&= x^2,\ 0 < x < 1 \\
&= 1,\ x \ge 1.
\end{aligned}
$$

Find the function $f(x)$.

Elementary Techniques of Calculus

You need to be able to integrate and differentiate the functions

$$x^n,\ \frac{1}{1+x},\ e^x,\ x^n e^x, \sin(x), \cos(x),\ \mathit{and}\ \ln(x)$$

and you must be conversant with the chain rule, the product rule, integration by parts, and substitution.

Problems Evaluate the indicated derivatives and integrals. a. $\frac{d}{dx}e^{x^2}$ b. $\frac{d}{dx}\frac{1}{(1+x)^2}$ c. $\int_0^1 xe^x\,dx$ d. $\int_0^\infty e^{-x}\,dx$ and e. $\int_0^\infty xe^{-x}\,dx$.

Problem We will make considerable use of the Gamma-function, defined for $z > 0$ by

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt.$$

Show using integration by parts that $\Gamma(z+1) = z\Gamma(z)$, and that if z is a positive integer, $\Gamma(z) = (z-1)!$.

Multivariable Integration: Examples

The ability to set up and evaluate integrals in 1 and 2 dimensions is essential. Here are two examples.

1. Integrate $f(x, y) = x^2y$ over the region $0 < x < 1$, $1 < y < 3$.

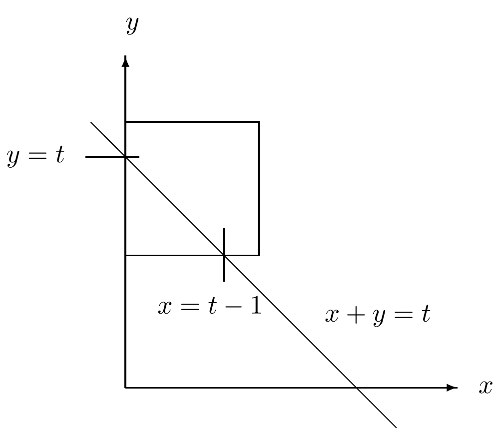
$$\int_1^3 \left(\int_0^1 x^2 y \, dx \right) dy \;=\; \int_1^3 \left(\frac{x^3}{3} y \, |_{x=0}^{x=1} dx \right) dy$$
$$= \int_1^3 [\frac{1\,y}{3} - \frac{0\,y}{3}]\, dy$$
$$= \frac{4}{3}.$$

2. **Tricky!** Evaluate the integral

$$I(t) = \int_{D \cap \{x+y \le t\}} xy \, dxdy$$

with $D = \{(x, y) : 0 < x < 1,\, 1 < y < 2\}$ and t any real number. Hint. DRAW a PICTURE for each of the cases $t \le 1$; $1 < t \le 2$; $2 < t \le 3$; $3 < t$ separately.

Answer: For example, for the case $1 < t \le 2$ we have the picture where the integral is over the triangle in the square



and so

$$I(t) = \int_0^{t-1} \int_1^{t-x} xy \, dydx = \int_1^t \int_0^{t-y} xy \, dxdy.$$

Doing this integral and the others, we find:

$$\begin{aligned} I(t) &= 0,\ t \le 1 \\ &= \frac{1}{2}[t^4/12 - t^2/2 + 2t/3 - 1/4],\ 1 < t \le 2 \\ &= \frac{1}{4}[-t^4/6 + 5t^2 - 12t + 15/2],\ 2 < t \le 3 \\ &= 2,\ 3 < t. \end{aligned}$$

Problem The reader is invited to perform the indicated integrals in the range $1 < t < 2$, and to set up and perform the integrals for $2 < t < 3$.