Week 08 notes

Functions on two random variables

01 Theory

PMF of any function of two variables

Suppose $W = g(X, Y)$ and X, Y are discrete RVs.

The PMF of W:

$$P_W(w) = \sum_{\substack{(x,y) \text{ s.t.} \\ g(x,y)=w}} P_{X,Y}(x, y)$$

i.e. sum over $g^{-1}(w)$

= "pre-image of w"

CDF of continuous function of two variables

Suppose $W = g(X, Y)$ and X, Y are continuous RVs, and g is a continuous function.

The CDF of W:

$$F_W(w) \;=\; P[W \le w] \;=\; \iint\limits_{g(x,y)\le w} f_{X,Y}(x, y)\, dx dy$$

$= g^{-1}(\{u \mid u \le w\})$

One can then compute the PDF of W by differentiation:

$$f_W(w) \;=\; \frac{d}{dw} F_W(w)$$

02 Illustration

Example - PDF of a quotient

$g(x,y) = y/x$

Suppose the joint PDF of X and Y is given by:

$$f_{X,Y}(x, y) = \begin{cases} \lambda\mu e^{-(\lambda x+\mu y)} & x,\, y \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

Find the PDF of $W = Y/X$.

1. Find the CDF using logic.
   - Convert to event form:

     $$= \iint_{y/x \le w} f_{X,Y}(x,y)\,dx\,dy$$

     $$F_W(w) \;=\; P[Y/X \le w]$$

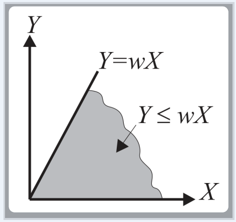
     "$y/x \le w$"

     $\Longleftrightarrow$ "$y \le wx$"

   - Re-express:

     $$\gg\gg \quad P[Y \le wX]$$

- Diagram:



- Compute:

$$
\begin{aligned}
P[Y \le wX] \quad &= \quad \int_{\cancel{0}\,-\infty}^{\infty} \int_{\cancel{0}\,-\infty}^{wx} f_{X,Y}(x,y)\,dy\,dx \\
\gg\gg \quad & \quad \int_0^{\infty} \lambda e^{-\lambda x} \int_0^{wx} \mu e^{-\mu y}\,dy\,dx \\
\gg\gg \quad & \quad \int_0^{\infty} \lambda e^{-\lambda x} \left(-e^{-\mu w x} + 1\right) dx \\
\gg\gg \quad & \quad 1 - \frac{\lambda}{\lambda + \mu w}
\end{aligned}
$$

2. Differentiate to find PDF.
   - Compute $\frac{d}{dw} F_W(w)$:

$\frac{dF}{dw} \cdot dw = dF$

i.e.

$f_W \cdot dw =$ "dP"

$$
f_W(w) = \begin{cases} \dfrac{\lambda\mu}{(\lambda+\mu w)^2} & w \ge 0 \\ 0 & \text{otherwise} \end{cases}
$$

Exercise - PMF of XY^2 from chart

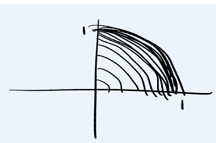
Suppose the joint PMF of X and Y is given by this chart:

$Y \downarrow \quad X \rightarrow$	1	2
-1	0.2	0.2
0	0.35	0.1
1	0.05	0.1

Define $W = XY^2$.

- (a) Find the PMF $P_W(w)$.
- (b) Find the expectation $E[W]$.

Example - Max and Min from joint PDF



Suppose the joint PDF of X and Y is given by:

$$
f_{X,Y}(x,y) \quad = \quad \begin{cases} \overbrace{\frac{3}{2}(x^2+y^2)}^{\frac{3}{2}R^2} & x,\ y \in [0,1] \\ 0 & \text{otherwise} \end{cases}
$$

Find the PDFs:

- (a) $W = \mathrm{Max}(X, Y)$
- (b) $W = \mathrm{Min}(X, Y)$

Solution

(a)

1. Compute CDF of W.
   - Convert to event form:

$$F_W(w) \quad = \quad P\big[\mathrm{Max}(X,Y) \le w\big]$$

   - Interpret:

$$\gg\gg \quad P\big[X \le w \text{ and } Y \le w\big]$$

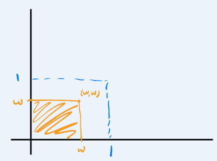
$P[\mathrm{Min}(X,Y) \le w] = ?$

$= P[X \le w \text{ OR } Y \le w]$

$= 1 - P[X > w \text{ and } Y > w]$

   - Integrate PDF over the region, assuming $w \in [0, 1]$:

$$\int_{-\infty}^{w} \int_{-\infty}^{w} f_{X,Y}(x, y)\, dx\, dy$$



   - Insert PDF formula:

$$\int_0^w \int_0^w \tfrac{3}{2}(x^2 + y^2)\, dx\, dy \quad \gg\gg \quad w^4$$

2. Differentiate to find $f_W(w)$.
   - $f_W = \frac{d}{dw} F_W(w)$:

$$f_W(w) \quad = \quad \begin{cases} 4w^3 & w \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

(b)

1. Compute CDF of W.
   - Convert to event form:

$$F_W(w) \;=\; P\big[\mathrm{Min}(X,Y) \le w\big]$$

   - Consider complement event to interpret:

$$\gg\gg \quad 1 - P\big[\mathrm{Min}(X,Y) > w\big] \quad \gg\gg \quad 1 - P[X > w \text{ and } Y > w]$$

   - Integrate PDF over the region:

$$P[X > w \text{ and } Y > w] \quad \gg\gg \quad \int_w^1 \int_w^1 \tfrac{3}{2}(x^2 + y^2)\, dx\, dy$$

   - Compute integral:

$$\gg\gg \quad w^4 - w^3 - w + 1$$

   - Therefore:

$$F_W(w) \;=\; w + w^3 - w^4$$

2. Differentiate to find $f_W(w)$.

- $f_W = \frac{d}{dw}F_W(w)$:

$$f_W(w) \quad = \quad \begin{cases} 1+3w^2-4w^3 & w \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Sums of random variables

03 Theory

The special case where $g(X,Y) = X+Y$ is very useful to study in further depth.

THEOREM: Continuous PDF of a sum

Suppose $f_{X,Y}(x,y)$ is the joint PDF for continuous RVs X and Y.

Then the PDF $f_W(w)$ of $W = X+Y$ is given by the formula:

$$f_W(w) \quad = \quad \int_{-\infty}^{+\infty} f_{X,Y}(w-x, x)\,dx$$

$\int_{x+y=w} f_{X,Y}$ "$d\ell$"

When X and Y are *independent*, so $f_{X,Y} = f_X f_Y$, the formula turns into the **convolution** of f_X and f_Y:

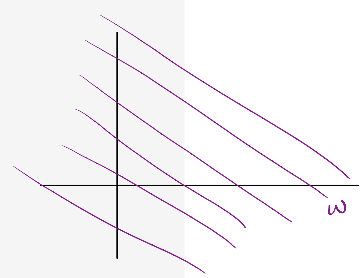
$$f_W(w) \quad = \quad f_X * f_Y \quad = \quad \int_{-\infty}^{+\infty} f_X(w-x)f_Y(x)\,dx$$

- There is no particular reason to choose the x-slot for $w-x$.
  - Equally valid to write: $f_W(w) \;=\; \int_{-\infty}^{+\infty} f_{X,Y}(x, w-x)\,dx$

Extra - Derivation of continuous PDF of a sum

The joint CDF of $X+Y$ is given by:

$$F_{X+Y}(w) = P[X+Y \leq w] \quad = \quad \iint_{x+y\leq w} f_{X,Y}(x,y)\,dx\,dy$$



From this we can find f_{X+Y} by taking the derivative:

$$f_{X+Y}(w) = \frac{d}{dw}F_{X+Y}(w) \quad \gg\gg \quad \frac{d}{dw}\iint_{x+y\leq w} f_{X,Y}(x,y)\,dx\,dy$$

In order to calculate this derivative, we change variables by setting $x = x$ and $s = x+y$. The Jacobian is 1, so $dx\,dy$ becomes $dx\,d\cancel{w}$ s, and we have:

$$\gg\gg \quad \frac{d}{dw}\int_{-\infty}^{w}\int_{-\infty}^{+\infty} f_{X,Y}(x, s-x)\,dx\,ds \quad \gg\gg \quad \int_{-\infty}^{+\infty} f_{X,Y}(x, \overbrace{w-x}^{=\text{"}y\text{"}})\,dx$$

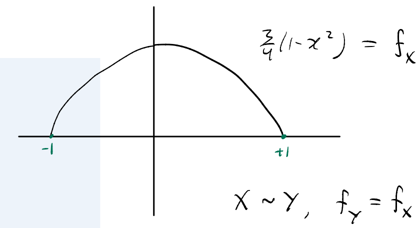
$dx\,dy \rightsquigarrow J \cdot dx\,ds$

04 Illustration

Example - Sum of parabolic random variables

Suppose X is an RV with PDF given by:

$$f_X(x) = \begin{cases} \frac{3}{4}(1-x^2) & x \in [-1,1] \\ 0 & \text{otherwise} \end{cases}$$

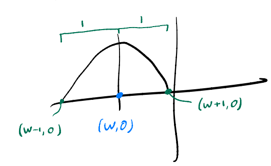


Let Y be an independent copy of X. So $f_Y = f_X$, but Y is independent of X.

Find the PDF of $X+Y$.

Solution

The graph of $f_X(w-x)$ matches the graph of $f_X(x)$ except (i) flipped in a vertical mirror, (ii) shifted by w to the left.



When $w \in [-2,0]$, the integrand is nonzero only for $x \in [-1, w+1]$:

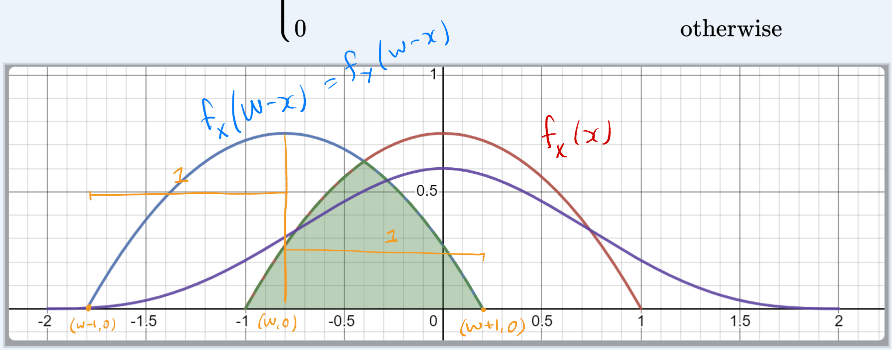
$$\begin{aligned} f_{X+Y}(w) &= \left(\frac{3}{4}\right)^2 \int_{-1}^{w+1} \left(1-(w-x)^2\right)\left(1-x^2\right) dx \\ &= \frac{9}{16}\left(\frac{w^5}{30} - \frac{2w^3}{3} - \frac{4w^2}{3} + \frac{16}{15}\right) \end{aligned}$$

When $w \in [0,+2]$, the integrand is nonzero only for $x \in [w-1,+1]$:

$$\begin{aligned} f_{X+Y}(w) &= \left(\frac{3}{4}\right)^2 \int_{w-1}^{+1} \left(1-(w-x)^2\right)\left(1-x^2\right) dx \\ &= \frac{9}{16}\left(-\frac{w^5}{30} + \frac{2w^3}{3} - \frac{4w^2}{3} + \frac{16}{15}\right) \end{aligned}$$

Final result is:

$$f_{X+Y}(w) = \begin{cases} \frac{9}{16}\left(\frac{w^5}{30} - \frac{2w^3}{3} - \frac{4w^2}{3} + \frac{16}{15}\right) & w \in [-2,0] \\ \frac{9}{16}\left(-\frac{w^5}{30} + \frac{2w^3}{3} - \frac{4w^2}{3} + \frac{16}{15}\right) & w \in [0,2] \\ 0 & \text{otherwise} \end{cases}$$



THEOREM: Discrete PMF of a sum

Suppose $P_{X,Y}(k,\ell)$ is the joint PMF for discrete RVs X and Y.

Assume that the possible value pairs are (k,ℓ) with $k, \ell \in \mathbb{Z}$ (integers only).

Then the PMF of $W = X+Y$ is given by the formula:

$\int f_X(w-x) f_Y(x)\, dx$

$j-i \;+\; i \;=\; j$

$$P_{X+Y}(j) = P_W(j) \quad = \quad \sum_{i=-\infty}^{+\infty} P_{X,Y}(j-i, i)$$

$$P_{X+Y}(w_j) = \sum_{\substack{k+\ell \\ = w_j}} P_{X,Y}(k,\ell)$$

PMF of $X+Y$ for discrete variables

Prove the discrete formula for the PMF of a sum.
(Apply the general formula for the PMF of $g(X, Y)$.)

05 Theory

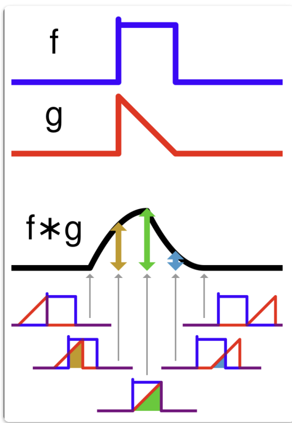
Convolution

The **convolution** of two continuous functions $f(x)$ and $g(x)$ is defined by:

$$(f * g)(x) \quad = \quad \int_{-\infty}^{+\infty} f(x-t)g(t)\, dt$$

$$= \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt$$

$$\text{"} = \int_{u+t=x} f(u)\, g(t)\, d? \text{"}$$



For more example calculations, look at 9.6.1 and 9.6.2 at [this page](#).

Applications of convolution

- Convolutional neural networks (machine learning theory: translation invariant NN, low pre-processing)
- Image processing: edge detection, blurring
- Signal processing: smoothing and interpolation estimation
- Electronics: linear translation-invariant (LTI) system response: convolution with impulse function

Extra - Convolution

Geometric meaning of convolution

Convolution does not have a neat and precise geometric meaning, but it does have an imprecise intuitive sense.

The product of two quantities tends to be large when *both* quantities are large; when one of them is small or zero, the product will be small or zero. This behavior is different from the behavior of a sum, where one summand being large is sufficient for the sum to be large. A large summand overrides a small co-summand, whereas a large factor is scaled down by a small cofactor.

The upshot is that a convolution will be large when two functions *have similar overall shape*. (Caveat: one function must be flipped in a vertical mirror before the overlay is considered.) The argument value where the convolution is largest will correspond to the horizontal offset needed to get the closest overlay of the functions.

Algebraic properties of convolution

- $f * g = g * f$
- $f * (g * h) = (f * g) * h$
- $f * (g + h) = f * g + f * h$
- $a(f * g) = (af) * g = f * (ag)$
- $(f * g)' = f' * g = f * g'$

The last of these is *not* the typical Leibniz rule for derivatives of products!

All of these properties can be checked by simple calculations with iterated integrals.

Convolution in more variables

Given $f, g : \mathbb{R}^n \to \mathbb{R}$, their convolution at $\mathbf{x}$ is defined by integrating the shifted products over the whole domain:

$$(f * g)(\mathbf{x}) = \iiint_{\mathbb{R}^n} f(\mathbf{x} - \mathbf{y}) g(\mathbf{y})\, dy$$

06 Illustration

Exercise - Convolution practice

- Suppose X is an RV with density:

$$f_X = \begin{cases} 2x & x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

- Suppose Y is uniform on $[0,1]$.

Find the PDF of $X + Y$. Sketch the graph of this PDF.

07 Theory

Some pairs of density functions have convolutions that can be described neatly in terms of the densities of known distributions, and sometimes this relationship has its own interpretation in the applied context of a probability model.

Bernoulli plus Binomial

Suppose $X_i \sim \text{Ber}(p)$ for $i = 1, 2, 3, \ldots$ are *independent* Bernoulli variables.

Define $S_n = X_1 + \cdots + X_n$, and notice that $S_n \sim \text{Bin}(n, p)$.

Then $S_n + X_{n+1} \sim S_{n+1}$ where $S_{n+1} \sim \text{Bin}(n+1, p)$.

In other words: adding a Bernoulli to a Binomial creates a bigger Binomial.

$\text{Ber}(p) + \ldots + \text{Ber}(p) = \text{Bin}(n,p)$

$\text{Ber}(p) + \text{Bin}(n,p) = \text{Bin}(n+1,p)$

Extra - Proof of Bernoulli sum rule

$\begin{cases} p & k=1 \\ (1-p) & k=0 \end{cases}$

For the PMF of X_{n+1}, we have $P_{X_{n+1}}(k) = p^k(1-p)^{1-k}$ for $k = 0, 1$, and $P_{X_{n+1}}(k) = 0$ for other k.

For the PMF of S_n we have $P_{S_n}(k) = \binom{n}{k} p^k (1-p)^{n-k}$ for $k = 0, \dots, n$ and $P_{S_n}(k) = 0$ for other k.

We seek the discrete convolution $(P_{S_n} * P_{X_{n+1}})(\ell)$.

The factor $P_{X_{n+1}}(\ell - k)$ in the convolution is nonzero only when $k = \ell$ or $k = \ell - 1$. So we have:

$$\begin{cases} p & \ell-k=1 \quad \text{i.e. } k=\ell-1 \\ 1-p & \ell-k=0 \quad \text{i.e. } k=\ell \\ 0 & \text{otherwise} \quad k = \text{else} \end{cases}$$

$$\begin{aligned}
(P_{S_n} * P_{X_{n+1}})(\ell) = \sum_{k=-\infty}^{+\infty} P_{S_n}(k) P_{X_{n}}(\ell - k) \\
&= P_{S_n}(\ell)(1-p) + P_{S_n}(\ell-1)p \\
&= \frac{n!}{\ell!(n-\ell)!} p^\ell (1-p)^{n-\ell+1} + \frac{n!}{(\ell-1)!(n-\ell+1)!} p^\ell (1-p)^{n-\ell+1} \\
&= \frac{n!(n-\ell+1) + n!\ell}{\ell!(n-\ell+1)!} p^\ell (1-p)^{n-\ell+1} \\
&= \frac{(n+1)!}{\ell!(n+1-\ell)!} p^\ell (1-p)^{n+1-\ell} \\
&= \binom{n+1}{\ell} p^\ell (1-p)^{n+1-\ell}
\end{aligned}$$

$\ell! \to \ell(\ell-1)!$

$(n-\ell+1)! = (n-\ell+1)(n-\ell)!$

This is the PMF of S_{n+1}, so we are done.

Binomial sum rule

Suppose $X \sim \text{Bin}(n, p)$ and $Y \sim \text{Bin}(m, p)$ are independent RVs with the given binomial distributions (same p, different numbers of trials).

Then $X + Y \sim \text{Bin}(n + m, p)$.

$\to \left(\underbrace{Ber(p) + \dots + Ber(p)}_{n}\right) + \left(\underbrace{Ber(p) + \dots + Ber(p)}_{m}\right) = \overbrace{Ber(p) + \dots + Ber(p)}^{n+m \text{ copies}}$

Extra - Proof of binomial sum rule

Of course, $X + Y$ measures the number of successes in $n + m$ independent trials, each with success probability p.

08 Illustration

Exercise - Vandermonde's identity from the binomial sum rule

Show that this "Vandermonde identity" holds for positive integers n, m, ℓ:

$$\sum_{j+k=\ell} \binom{n}{j} \binom{m}{k} = \binom{n+m}{\ell}$$

Hint: The binomial sum rule is:

$$\text{“Bin}(n,p) + \text{Bin}(m,p) \sim \text{Bin}(n+m,p)\text{”}$$

Set $p = q = 1/2$. Compute the PMF of the left side using convolution. Compute the PMF of the right side directly. Set these PMFs equal.

09 Theory

Recall that a Poisson variable counts 'arrivals' in a fixed time window. It applies to events like phone calls per hour or Uranium decays per second. Each interval is independent of the others, and the rate of occurrences is proportional to the size of the interval.

An implication of this meaning of the Poisson variable is a sum rule. If you divide a Poisson interval into subintervals, the distribution corresponding to each subinterval should still be Poisson, and the distribution of arrivals in each subinterval should *add up* to give the distribution of arrivals for the total interval.

Poisson sum rule

Suppose $X \sim \text{Pois}(\lambda)$ and $Y \sim \text{Pois}(\mu)$ and X and Y are independent.

Then $X + Y \sim \text{Pois}(\lambda + \mu)$.

Extra - Proof of Poisson sum rule

Write $p_X(k) = e^{-\lambda}\frac{\lambda^k}{k!}$ and $p_Y(k) = e^{-\mu}\frac{\mu^k}{k!}$. Then:

$$\begin{aligned} P_{X+Y}(n) &= P[X+Y=n] \\ &= \sum_{k=-\infty}^{+\infty} P_X(k)P_Y(n-k) \\ &= \sum_{k=0}^{n} e^{-(\lambda+\mu)}\frac{\lambda^k\mu^{n-k}}{k!(n-k)!} \\ &= \frac{e^{-(\lambda+\mu)}}{n!}\sum_{k=0}^{n}\binom{n}{k}\lambda^k\mu^{n-k} \\ &= e^{-(\lambda+\mu)}\frac{(\lambda+\mu)^n}{n!} \end{aligned}$$

Recall that in a Bernoulli process:

- An RV measuring the discrete wait time until one success has a geometric distribution.
- An RV measuring discrete wait time until ℓ^{th} success has a Pascal distribution.

Since the wait times between successes are *independent*, we expect that the *sum of geometric distributions is a Pascal distribution*. This is true!

Pascal Sum Rule

Specify a given Bernoulli process with success probability p. Suppose that:

- $X \sim \text{Pascal}(r, p)$
- $Y \sim \text{Pascal}(s, p)$
- X and Y are independent

Then $X + Y \sim \text{Pascal}(r + s, p)$.

Geom plus Geom is Pascal

Recall that $\text{Pascal}(1, p) \sim \text{Geom}(p)$. So we could say:

$$\text{“Geom}(p) + \text{Geom}(p) \ = \ \text{Pascal}(2, p)\text{”}$$

And:

$$\text{“Geom}(p) + \text{Pascal}(r, p) \ = \ \text{Pascal}(r + 1, p)\text{”}$$

$\underbrace{\text{Geom}(p) + \dots + \text{Geom}(p)}_{(r+1)\text{-times}}$

The Pascal Sum Rule can be justified in two ways:

- (1) by directly computing the discrete convolution of two Pascal variables
- (2) by observing that the sum $X + Y$ counts the trials until exactly $r + s$ successes
  - Waiting for r successes and then waiting for s successes is the same as waiting for $r + s$ successes

---

Recall that in a Poisson process:

- An RV measuring continuous wait time until one arrival has an exponential distribution.
- An RV measuring continuous wait time until ℓ^{th} arrival has an Erlang distribution.

Since the wait times between arrivals are *independent*, we expect that the *sum of exponential distributions is an Erlang distribution*. This is true!

Erlang sum rule

Specify a given Bernoulli process with success probability p. Suppose that:

- $X \sim \text{Erlang}(r, \lambda)$
- $Y \sim \text{Erlang}(s, \lambda)$
- X and Y are independent

Then $X + Y \sim \text{Erlang}(r + s, \lambda)$.

Exp plus Exp is Erlang

Recall that $\text{Erlang}(1, \lambda) \sim \text{Exp}(\lambda)$. So we could say:

$$\text{“Exp}(\lambda) + \text{Exp}(\lambda) \ = \ \text{Erlang}(2, \lambda)\text{”}$$

And:

$$\text{“Exp}(\lambda) + \text{Erlang}(\ell, \lambda) \quad = \quad \text{Erlang}(\ell + 1, \lambda)\text{”}$$

10 Illustration

Example - Exp plus Exp equals Erlang

Let us verify this formula by direct calculation:

$$\text{“Exp}(\lambda) + \text{Exp}(\lambda) \quad = \quad \text{Erlang}(2, \lambda)\text{”}$$

Solution

Let X, $Y \sim \text{Exp}(\lambda)$ be independent RVs.

Therefore:

$$f_X = f_Y \quad = \quad \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Now compute the convolution:

$$f_{X+Y}(w) \quad = \quad \int_{-\infty}^{+\infty} f_X(w - x) f_Y(x)\, dx$$

$$\gg\gg \quad \int_0^w \lambda^2 e^{-\lambda(w-x)} e^{-\lambda x}\, dx \quad \gg\gg \quad \lambda^2 \int_0^w e^{-\lambda w}\, dx \quad \gg\gg \quad \lambda^2 w e^{-\lambda w}$$

This is the Erlang PDF:

for $\ell = 2$

$$f_X(t) = \frac{\lambda^\ell}{(\ell - 1)!} t^{\ell - 1} e^{-\lambda t} \Big|_{\ell = 2}$$

Exercise - Erlang induction step

By direct computation with PDFs and convolution, derive the formula:

$$\text{“Exp}(\lambda) + \text{Erlang}(\ell, \lambda) \quad = \quad \text{Erlang}(\ell + 1, \lambda)\text{”}$$

By repeated iteration of the above formula, starting with $\ell = 1$, one can derive the PMF for any Erlang variable as the sum of exponential variables:

$$\text{“}\overbrace{\text{Exp}(\lambda) + \cdots + \text{Exp}(\lambda)}^{\ell \text{ terms}} \quad = \quad \text{Erlang}(\ell, \lambda)\text{”}$$

This fully explains the formula for the Erlang PDF.

11 Theory

Normal sum rule

Suppose we know:

- $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$

- $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$
- X and Y are independent

Then:

$$X + Y \ \sim\ \mathcal{N}\big(\mu_X + \mu_Y,\ \sigma_X^2 + \sigma_Y^2\big)$$

Recall that $aX + b$ is normal if X is normal; more specifically $aX + b \sim \mathcal{N}(a\mu_X + b, a^2\sigma_X^2)$ when $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$.

This fact, combined with the sum rule, implies that $W = aX + bY + c$ is normal when X and Y are *independent* normals. Then $E[W]$ and $\mathrm{Var}[W]$ are easily computed using the linearity rules:

$$\mu_W = a\,\mu_X + b\,\mu_Y + c, \quad \sigma_W^2 = (a\,\sigma_X)^2 + (b\,\sigma_Y)^2$$

12 Illustration

Combining normals

Suppose $X \sim \mathcal{N}(40, 16)$, $Y \sim \mathcal{N}(15, 9)$. Find the probability that $X \geq 2Y$.

Solution
Define $W = X - 2Y$. Using the formulas above, we see $W \sim \mathcal{N}(10, 52)$, or $W \sim \sqrt{52}Z + 10$ for a standard normal Z. Then:

$$P[X \geq 2Y] \quad \gg\gg \quad P[W \geq 0] \quad \gg\gg \quad P\left[Z \geq \tfrac{-10}{\sqrt{52}}\right]$$

$$\gg\gg \quad P[Z \leq 1.39] \quad \gg\gg \quad \approx 0.918$$