Probability and Stochastic Processes:
A Friendly Introduction for Electrical and Computer Engineers
Edition 3

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Yates and Goodman 3e Solution Set: 9.1.1, 9.1.2, 9.1.3, 9.1.5, 9.5.1, 9.5.2, 9.5.4, and 9.5.7

Problem 9.1.1 Solution

Let $Y = X_1 - X_2$.

(a) Since $Y = X_1 + (-X_2)$, Theorem 9.1 says that the expected value of the difference is

$$\mathrm{E}[Y] = \mathrm{E}[X_1] + \mathrm{E}[-X_2] = \mathrm{E}[X] - \mathrm{E}[X] = 0. \tag{1}$$

(b) By Theorem 9.2, the variance of the difference is

$$\mathrm{Var}[Y] = \mathrm{Var}[X_1] + \mathrm{Var}[-X_2] = 2\,\mathrm{Var}[X]. \tag{2}$$

Problem 9.1.2 Solution

The random variable X_{33} is a Bernoulli random variable that indicates the result of flip 33. The PMF of X_{33} is

$$P_{X_{33}}(x) = \begin{cases} 1-p & x=0, \\ p & x=1, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Note that each X_i has expected value $\mathrm{E}[X] = p$ and variance $\mathrm{Var}[X] = p(1-p)$. The random variable $Y = X_1 + \cdots + X_{100}$ is the number of heads in 100 coin flips. Hence, Y has the binomial PMF

$$P_Y(y) = \begin{cases} \binom{100}{y} p^y (1-p)^{100-y} & y = 0, 1, \ldots, 100, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Since the X_i are independent, by Theorems 9.1 and 9.3, the mean and variance of Y are

$$\mathrm{E}[Y] = 100\,\mathrm{E}[X] = 100p, \qquad \mathrm{Var}[Y] = 100\,\mathrm{Var}[X] = 100p(1-p). \tag{3}$$

Problem 9.1.3 Solution

(a) The PMF of N_1, the number of phone calls needed to obtain the correct answer, can be determined by observing that if the correct answer is given on the nth call, then the previous $n-1$ calls must have given wrong answers so that

$$P_{N_1}(n) = \begin{cases} (3/4)^{n-1}(1/4) & n = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(b) N_1 is a geometric random variable with parameter $p = 1/4$. In Theorem 3.5, the mean of a geometric random variable is found to be $1/p$. For our case, $E[N_1] = 4$.

(c) Using the same logic as in part (a) we recognize that in order for n to be the fourth correct answer, that the previous $n-1$ calls must have contained exactly 3 correct answers and that the fourth correct answer arrived on the n-th call. This is described by a Pascal random variable.

$$P_{N_4}(n_4) = \begin{cases} \binom{n-1}{3}(3/4)^{n-4}(1/4)^4 & n = 4, 5, \ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

(d) Using the hint given in the problem statement we can find the mean of N_4 by summing up the means of the 4 identically distributed geometric random variables each with mean 4. This gives $E[N_4] = 4\,E[N_1] = 16$.

Problem 9.1.5 Solution

We can solve this problem using Theorem 9.2 which says that

$$\text{Var}[W] = \text{Var}[X] + \text{Var}[Y] + 2\,\text{Cov}\,[X, Y]\,. \tag{1}$$

The first two moments of X are

$$E\,[X] = \int_0^1 \int_0^{1-x} 2x\,dy\,dx = \int_0^1 2x(1-x)\,dx = 1/3, \tag{2}$$

$$E\left[X^2\right] = \int_0^1 \int_0^{1-x} 2x^2\,dy\,dx = \int_0^1 2x^2(1-x)\,dx = 1/6. \tag{3}$$

Thus the variance of X is $\text{Var}[X] = \text{E}[X^2] - (\text{E}[X])^2 = 1/18$. By symmetry, it should be apparent that $\text{E}[Y] = \text{E}[X] = 1/3$ and $\text{Var}[Y] = \text{Var}[X] = 1/18$. To find the covariance, we first find the correlation

$$\text{E}[XY] = \int_0^1 \int_0^{1-x} 2xy\, dy\, dx = \int_0^1 x(1-x)^2\, dx = 1/12. \tag{4}$$

The covariance is

$$\text{Cov}[X,Y] = \text{E}[XY] - \text{E}[X]\,\text{E}[Y] = 1/12 - (1/3)^2 = -1/36. \tag{5}$$

Finally, the variance of the sum $W = X + Y$ is

$$\begin{aligned} \text{Var}[W] &= \text{Var}[X] + \text{Var}[Y] - 2\,\text{Cov}[X,Y] \\ &= 2/18 - 2/36 = 1/18. \end{aligned} \tag{6}$$

For this specific problem, it's arguable whether it would easier to find $\text{Var}[W]$ by first deriving the CDF and PDF of W. In particular, for $0 \le w \le 1$,

$$\begin{aligned} F_W(w) &= \text{P}[X + Y \le w] \\ &= \int_0^w \int_0^{w-x} 2\, dy\, dx \\ &= \int_0^w 2(w-x)\, dx = w^2. \end{aligned} \tag{7}$$

Hence, by taking the derivative of the CDF, the PDF of W is

$$f_W(w) = \begin{cases} 2w & 0 \le w \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

From the PDF, the first and second moments of W are

$$\text{E}[W] = \int_0^1 2w^2\, dw = 2/3, \qquad \text{E}\left[W^2\right] = \int_0^1 2w^3\, dw = 1/2. \tag{9}$$

The variance of W is $\text{Var}[W] = \text{E}[W^2] - (\text{E}[W])^2 = 1/18$. Not surprisingly, we get the same answer both ways.

Problem 9.5.1 Solution

We know that the waiting time, W is uniformly distributed on [0,10] and therefore has the following PDF.

$$f_W(w) = \begin{cases} 1/10 & 0 \leq w \leq 10, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

We also know that the total time is 3 milliseconds plus the waiting time, that is $X = W + 3$.

(a) The expected value of X is $E[X] = E[W+3] = E[W] + 3 = 5 + 3 = 8$.

(b) The variance of X is $\text{Var}[X] = \text{Var}[W+3] = \text{Var}[W] = 25/3$.

(c) The expected value of A is $E[A] = 12\,E[X] = 96$.

(d) The standard deviation of A is $\sigma_A = \sqrt{\text{Var}[A]} = \sqrt{12(25/3)} = 10$.

(e) $P[A > 116] = 1 - \Phi(\frac{116-96}{10}) = 1 - \Phi(2) = 0.02275$.

(f) $P[A < 86] = \Phi(\frac{86-96}{10}) = \Phi(-1) = 1 - \Phi(1) = 0.1587$.

Problem 9.5.4 Solution

In Theorem 9.7, we learned that a sum of iid Poisson random variables is a Poisson random variable. Hence W_n is a Poisson random variable with mean $\mathrm{E}[W_n] = n\,\mathrm{E}[K] = n$. Thus W_n has variance $\mathrm{Var}[W_n] = n$ and PMF

$$P_{W_n}(w) = \begin{cases} n^w e^{-n}/w! & w = 0, 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

All of this implies that we can exactly calculate

$$\mathrm{P}[W_n = n] = P_{W_n}(n) = n^n e^{-n}/n! \tag{2}$$

Since we can perform the exact calculation, using a central limit theorem may seem silly; however for large n, calculating n^n or $n!$ is difficult for large n. Moreover, it's interesting to see how good the approximation is. In this case, the approximation is

$$\begin{aligned} \mathrm{P}[W_n = n] &= \mathrm{P}[n \le W_n \le n] \\ &\approx \Phi\left(\frac{n + 0.5 - n}{\sqrt{n}}\right) - \Phi\left(\frac{n - 0.5 - n}{\sqrt{n}}\right) \\ &= 2\Phi\left(\frac{1}{2\sqrt{n}}\right) - 1. \end{aligned} \tag{3}$$

The comparison of the exact calculation and the approximation are given in the following table.

$P[W_n = n]$	$n = 1$	$n = 4$	$n = 16$	$n = 64$
exact	0.3679	0.1954	0.0992	0.0498
approximate	0.3829	0.1974	0.0995	0.0498

(4)

Problem 9.5.7 Solution

Random variable K_n has a binomial distribution for n trials and success probability $P[V] = 3/4$.

(a) The expected number of video packets out of 48 packets is

$$E[K_{48}] = 48\,P[V] = 36. \tag{1}$$

(b) The variance of K_{48} is

$$\text{Var}[K_{48}] = 48\,P[V](1 - P[V]) = 48(3/4)(1/4) = 9 \tag{2}$$

Thus K_{48} has standard deviation $\sigma_{K_{48}} = 3$.

(c) Using the ordinary central limit theorem and Table 4.2 yields

$$\begin{aligned} P[30 \le K_{48} \le 42] &\approx \Phi\left(\frac{42-36}{3}\right) - \Phi\left(\frac{30-36}{3}\right) \\ &= \Phi(2) - \Phi(-2) \end{aligned} \tag{3}$$

Recalling that $\Phi(-x) = 1 - \Phi(x)$, we have

$$P[30 \le K_{48} \le 42] \approx 2\Phi(2) - 1 = 0.9545. \tag{4}$$

(d) Since K_{48} is a discrete random variable, we can use the De Moivre-Laplace approximation to estimate

$$\begin{aligned} P[30 \le K_{48} \le 42] &\approx \Phi\left(\frac{42+0.5-36}{3}\right) - \Phi\left(\frac{30-0.5-36}{3}\right) \\ &= 2\Phi(2.16666) - 1 = 0.9687. \end{aligned} \tag{5}$$