Probability and Stochastic Processes:
A Friendly Introduction for Electrical and Computer Engineers
Edition 3
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Yates and Goodman 3e Solution Set: 7.1.1, 7.1.3, 7.1.7, 7.2.2, 7.2.3, 7.2.4, 7.2.6, 7.2.7, 7.4.4, 7.4.5, 7.4.6, 7.5.1, and 7.5.3

Problem 7.1.1 Solution

Given the CDF $F_X(x)$, we can write

$$\begin{aligned} F_{X|X>0}(x) &= \mathrm{P}\left[X \leq x | X > 0\right] \\ &= \frac{\mathrm{P}\left[X \leq x, X > 0\right]}{\mathrm{P}\left[X > 0\right]} \\ &= \frac{\mathrm{P}\left[0 < X \leq x\right]}{\mathrm{P}\left[X > 0\right]} = \begin{cases} 0 & x \leq 0, \\ \frac{F_X(x)-F_X(0)}{\mathrm{P}[X>0]}. & x > 0. \end{cases} \end{aligned} \tag{1}$$

From $F_X(x)$, we know that $F_X(0) = 0.4$ and $\mathrm{P}[X > 0] = 1 - F_X(0) = 0.6$. Thus

$$\begin{aligned} F_{X|X>0}(x) &= \begin{cases} 0 & x \leq 0, \\ \frac{F_X(x)-0.4}{0.6} & x > 0, \end{cases} \\ &= \begin{cases} 0 & x < 5, \\ \frac{0.8-0.4}{0.6} = \frac{2}{3} & 5 \leq x < 7, \\ 1 & x \geq 7. \end{cases} \end{aligned} \tag{2}$$

From the jumps in the conditional CDF $F_{X|X>0}(x)$, we can write down the conditional PMF

$$P_{X|X>0}(x) = \begin{cases} 2/3 & x = 5, \\ 1/3 & x = 7, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Alternatively, we can start with the jumps in $F_X(x)$ and read off the PMF of X as

$$P_X(x) = \begin{cases} 0.4 & x = -3, \\ 0.4 & x = 5, \\ 0.2 & x = 7, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

The event $\{X > 0\}$ has probability $\mathrm{P}[X > 0] = P_X(5) + P_X(7) = 0.6$. From Theorem 7.1, the conditional PMF of X given $X > 0$ is

$$P_{X|X>0}(x) = \begin{cases} \frac{P_X(x)}{\mathrm{P}[X>0]} & x \in B, \\ 0 & \text{otherwise,} \end{cases} = \begin{cases} 2/3 & x = 5, \\ 1/3 & x = 7, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Problem 7.1.3 Solution

The event $B = \{X \neq 0\}$ has probability $\mathrm{P}[B] = 1 - \mathrm{P}[X = 0] = 15/16$. The conditional PMF of X given B is

$$P_{X|B}(x) = \begin{cases} \frac{P_X(x)}{\mathrm{P}[B]} & x \in B, \\ 0 & \text{otherwise,} \end{cases} = \binom{4}{x} \frac{1}{15}. \tag{1}$$

Problem 7.1.7 Solution

(a) Given that a person is healthy, X is a Gaussian ($\mu = 90, \sigma = 20$) random variable. Thus,

$$f_{X|H}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} = \frac{1}{20\sqrt{2\pi}} e^{-(x-90)^2/800}. \tag{1}$$

(b) Given the event H, we use the conditional PDF $f_{X|H}(x)$ to calculate the required probabilities

$$\begin{aligned} \mathrm{P}\left[T^+|H\right] = \mathrm{P}\left[X \geq 140|H\right] &= \mathrm{P}\left[X - 90 \geq 50|H\right] \\ &= \mathrm{P}\left[\frac{X-90}{20} \geq 2.5|H\right] \\ &= 1 - \Phi(2.5) = 0.006. \end{aligned} \tag{2}$$

Similarly,

$$\begin{aligned} \mathrm{P}\left[T^{-}|H\right] = \mathrm{P}\left[X \leq 110|H\right] &= \mathrm{P}\left[X - 90 \leq 20|H\right] \\ &= \mathrm{P}\left[\frac{X-90}{20} \leq 1|H\right] \\ &= \Phi(1) = 0.841. \end{aligned} \tag{3}$$

(c) Using Bayes Theorem, we have

$$\mathrm{P}\left[H|T^{-}\right] = \frac{\mathrm{P}\left[T^{-}|H\right]\mathrm{P}\left[H\right]}{\mathrm{P}\left[T^{-}\right]} = \frac{\mathrm{P}\left[T^{-}|H\right]\mathrm{P}\left[H\right]}{\mathrm{P}\left[T^{-}|D\right]\mathrm{P}\left[D\right] + \mathrm{P}\left[T^{-}|H\right]\mathrm{P}\left[H\right]}. \tag{4}$$

In the denominator, we need to calculate

$$\begin{aligned} \mathrm{P}\left[T^{-}|D\right] = \mathrm{P}\left[X \leq 110|D\right] &= \mathrm{P}\left[X - 160 \leq -50|D\right] \\ &= \mathrm{P}\left[\frac{X-160}{40} \leq -1.25|D\right] \\ &= \Phi(-1.25) = 1 - \Phi(1.25) = 0.106. \end{aligned} \tag{5}$$

Thus,

$$\begin{aligned} \mathrm{P}\left[H|T^{-}\right] &= \frac{\mathrm{P}\left[T^{-}|H\right]\mathrm{P}\left[H\right]}{\mathrm{P}\left[T^{-}|D\right]\mathrm{P}\left[D\right] + \mathrm{P}\left[T^{-}|H\right]\mathrm{P}\left[H\right]} \\ &= \frac{0.841(0.9)}{0.106(0.1) + 0.841(0.9)} = 0.986. \end{aligned} \tag{6}$$

(d) Since T^{-}, T^{0}, and T^{+} are mutually exclusive and collectively exhaustive,

$$\begin{aligned} \mathrm{P}\left[T^{0}|H\right] &= 1 - \mathrm{P}\left[T^{-}|H\right] - \mathrm{P}\left[T^{+}|H\right] \\ &= 1 - 0.841 - 0.006 = 0.153. \end{aligned} \tag{7}$$

We say that a test is a failure if the result is T^{0}. Thus, given the event H, each test has conditional failure probability of $q = 0.153$, or success probability $p = 1 - q = 0.847$. Given H, the number of trials N until a success is a geometric (p) random variable with PMF

$$P_{N|H}(n) = \begin{cases} (1-p)^{n-1}p & n = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Problem 7.2.2 Solution

From the solution to Problem 3.4.2, the PMF of X is

$$P_X(x) = \begin{cases} 0.2 & x = -1, \\ 0.5 & x = 0, \\ 0.3 & x = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The event $B = \{|X| > 0\}$ has probability $P[B] = P[X \neq 0] = 0.5$. From Theorem 7.1, the conditional PMF of X given B is

$$P_{X|B}(x) = \begin{cases} \frac{P_X(x)}{P[B]} & x \in B, \\ 0 & \text{otherwise,} \end{cases} = \begin{cases} 0.4 & x = -1, \\ 0.6 & x = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

The conditional first and second moments of X are

$$E[X|B] = \sum_x x P_{X|B}(x) = (-1)(0.4) + 1(0.6) = 0.2, \tag{3}$$

$$E\left[X^2|B\right] = \sum_x x^2 P_{X|B}(x) = (-1)^2(0.4) + 1^2(0.6) = 1. \tag{4}$$

The conditional variance of X is

$$\text{Var}[X|B] = E\left[X^2|B\right] - (E[X|B])^2 = 1 - (0.2)^2 = 0.96. \tag{5}$$

Problem 7.2.3 Solution

The PDF of X is

$$f_X(x) = \begin{cases} 1/10 & -5 \leq x \leq 5, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(a) The event B has probability

$$P[B] = P[-3 \leq X \leq 3] = \int_{-3}^{3} \frac{1}{10}\, dx = \frac{3}{5}. \tag{2}$$

From Definition 7.3, the conditional PDF of X given B is

$$f_{X|B}(x) = \begin{cases} f_X(x) / \mathrm{P}[B] & x \in B, \\ 0 & \text{otherwise}, \end{cases} = \begin{cases} 1/6 & |x| \leq 3, \\ 0 & \text{otherwise}. \end{cases} \tag{3}$$

(b) Given B, we see that X has a uniform PDF over $[a, b]$ with $a = -3$ and $b = 3$. From Theorem 4.6, the conditional expected value of X is $\mathrm{E}[X|B] = (a+b)/2 = 0$.

(c) From Theorem 4.6, the conditional variance of X is $\mathrm{Var}[X|B] = (b-a)^2/12 = 3$.

Problem 7.2.4 Solution

From Definition 4.6, the PDF of Y is

$$f_Y(y) = \begin{cases} (1/5)e^{-y/5} & y \geq 0, \\ 0 & \text{otherwise}. \end{cases} \tag{1}$$

(a) The event A has probability

$$\begin{aligned} \mathrm{P}[A] = \mathrm{P}[Y < 2] &= \int_0^2 (1/5)e^{-y/5}\, dy \\ &= -e^{-y/5}\Big|_0^2 = 1 - e^{-2/5}. \end{aligned} \tag{2}$$

From Definition 7.3, the conditional PDF of Y given A is

$$\begin{aligned} f_{Y|A}(y) &= \begin{cases} f_Y(y) / \mathrm{P}[A] & x \in A, \\ 0 & \text{otherwise}, \end{cases} \\ &= \begin{cases} (1/5)e^{-y/5}/(1 - e^{-2/5}) & 0 \leq y < 2, \\ 0 & \text{otherwise}. \end{cases} \end{aligned} \tag{3}$$

(b) The conditional expected value of Y given A is

$$\mathrm{E}[Y|A] = \int_{-\infty}^{\infty} y f_{Y|A}(y)\, dy = \frac{1/5}{1 - e^{-2/5}} \int_0^2 y e^{-y/5}\, dy. \tag{4}$$

Using the integration by parts formula $\int u\,dv = uv - \int v\,du$ with $u = y$ and $dv = e^{-y/5}\,dy$ yields

$$\begin{aligned} \mathrm{E}[Y|A] &= \frac{1/5}{1-e^{-2/5}}\left(-5ye^{-y/5}\Big|_0^2 + \int_0^2 5e^{-y/5}\,dy \right) \\ &= \frac{1/5}{1-e^{-2/5}}\left(-10e^{-2/5} - 25e^{-y/5}\Big|_0^2 \right) \\ &= \frac{5-7e^{-2/5}}{1-e^{-2/5}}. \end{aligned} \tag{5}$$

Problem 7.2.6 Solution

Recall that the PMF of the number of pages in a fax is

$$P_X(x) = \begin{cases} 0.15 & x = 1,2,3,4, \\ 0.1 & x = 5,6,7,8, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(a) The event that a fax was sent to machine A can be expressed mathematically as the event that the number of pages X is an even number. Similarly, the event that a fax was sent to B is the event that X is an odd number. Since $S_X = \{1, 2, \ldots, 8\}$, we define the set $A = \{2, 4, 6, 8\}$. Using this definition for A, we have that the event that a fax is sent to A is equivalent to the event $X \in A$. The event A has probability

$$\mathrm{P}[A] = P_X(2) + P_X(4) + P_X(6) + P_X(8) = 0.5. \tag{2}$$

Given the event A, the conditional PMF of X is

$$P_{X|A}(x) = \begin{cases} \frac{P_X(x)}{\mathrm{P}[A]} & x \in A, \\ 0 & \text{otherwise,} \end{cases} = \begin{cases} 0.3 & x = 2,4, \\ 0.2 & x = 6,8, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

The conditional first and second moments of X given A is

$$\begin{aligned}
\mathrm{E}[X|A] &= \sum_x x P_{X|A}(x) \\
&= 2(0.3) + 4(0.3) + 6(0.2) + 8(0.2) = 4.6, \qquad (4)\\
\mathrm{E}\left[X^2|A\right] &= \sum_x x^2 P_{X|A}(x) \\
&= 4(0.3) + 16(0.3) + 36(0.2) + 64(0.2) = 26. \qquad (5)
\end{aligned}$$

The conditional variance and standard deviation are

$$\begin{aligned}
\mathrm{Var}[X|A] &= \mathrm{E}\left[X^2|A\right] - (\mathrm{E}[X|A])^2 = 26 - (4.6)^2 = 4.84, \qquad (6)\\
\sigma_{X|A} &= \sqrt{\mathrm{Var}[X|A]} = 2.2. \qquad (7)
\end{aligned}$$

(b) Let the event B' denote the event that the fax was sent to B and that the fax had no more than 6 pages. Hence, the event $B' = \{1, 3, 5\}$ has probability

$$\mathrm{P}\left[B'\right] = P_X(1) + P_X(3) + P_X(5) = 0.4. \qquad (8)$$

The conditional PMF of X given B' is

$$P_{X|B'}(x) = \begin{cases} \frac{P_X(x)}{\mathrm{P}[B']} & x \in B', \\ 0 & \text{otherwise}, \end{cases} = \begin{cases} 3/8 & x = 1, 3, \\ 1/4 & x = 5, \\ 0 & \text{otherwise}. \end{cases} \qquad (9)$$

Given the event B', the conditional first and second moments are

$$\begin{aligned}
\mathrm{E}\left[X|B'\right] &= \sum_x x P_{X|B'}(x) \\
&= 1(3/8) + 3(3/8) + 5(1/4) + = 11/4, \qquad (10)\\
\mathrm{E}\left[X^2|B'\right] &= \sum_x x^2 P_{X|B'}(x) \\
&= 1(3/8) + 9(3/8) + 25(1/4) = 10. \qquad (11)
\end{aligned}$$

The conditional variance and standard deviation are

$$\begin{aligned}
\mathrm{Var}[X|B'] &= \mathrm{E}\left[X^2|B'\right] - (\mathrm{E}\left[X|B'\right])^2 = 10 - (11/4)^2 = 39/16, \qquad (12)\\
\sigma_{X|B'} &= \sqrt{\mathrm{Var}[X|B']} = \sqrt{39}/4 \approx 1.56. \qquad (13)
\end{aligned}$$

Problem 7.2.7 Solution

(a) Consider each circuit test as a Bernoulli trial such that a failed circuit is called a success. The number of trials until the first success (i.e. a failed circuit) has the geometric PMF

$$P_N(n) = \begin{cases} (1-p)^{n-1}p & n = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(b) The probability there are at least 20 tests is

$$\mathrm{P}[B] = \mathrm{P}[N \geq 20] = \sum_{n=20}^{\infty} P_N(n) = (1-p)^{19}. \tag{2}$$

Note that $(1-p)^{19}$ is just the probability that the first 19 circuits pass the test, which is what we would expect since there must be at least 20 tests if the first 19 circuits pass. The conditional PMF of N given B is

$$\begin{aligned} P_{N|B}(n) &= \begin{cases} \frac{P_N(n)}{\mathrm{P}[B]} & n \in B, \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} (1-p)^{n-20}p & n = 20, 21, \ldots, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{3}$$

(c) Given the event B, the conditional expectation of N is

$$\mathrm{E}[N|B] = \sum_n n P_{N|B}(n) = \sum_{n=20}^{\infty} n(1-p)^{n-20}p. \tag{4}$$

Making the substitution $j = n - 19$ yields

$$\mathrm{E}[N|B] = \sum_{j=1}^{\infty} (j+19)(1-p)^{j-1}p = 1/p + 19. \tag{5}$$

We see that in the above sum, we effectively have the expected value of $J + 19$ where J is geometric random variable with parameter p. This is not surprising since the $N \geq 20$ iff we observed 19 successful tests. After 19 successful tests, the number of additional tests needed to find the first failure is still a geometric random variable with mean $1/p$.

Problem 7.4.4 Solution

Given $X = x$, $Y = x + Z$. Since Z is independent of X, we see that conditioned on $X = x$, Y is the sum of a constant x and additive Gaussian noise Z with standard deviation 1. That is, given $X = x$, Y is a Gaussian $(x, 1)$ random variable. The conditional PDF of Y is

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}} e^{-(y-x)^2/2}. \tag{1}$$

If this argument is unconvincing, we observe that the conditional CDF of Y given $X = x$ is

$$\begin{aligned} F_{Y|X}(y|x) &= \mathrm{P}[Y \leq y|X = x] \\ &= \mathrm{P}[x + Z \leq y|X = x] \\ &= \mathrm{P}[x + Z \leq y] \\ &= \mathrm{P}[Z \leq y - x] \\ &= F_Z(y - x), \end{aligned} \tag{2}$$

where the key step is that $X = x$ conditioning is removed in (2) because Z is independent of X. Taking a derivative with respect to y, we obtain

$$f_{Y|X}(y|x) = \frac{dF_{Y|X}(y|x)}{dy} = \frac{dF_Z(y - x)}{dy} = f_Z(y - x). \tag{3}$$

Since Z is Gaussian $(0, 1)$, the conditional PDF of Y is

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}} e^{-(y-x)^2/2}. \tag{4}$$

Problem 7.4.5 Solution

The main part of this problem is just interpreting the problem statement. No calculations are necessary. Since a trip is equally likely to last 2, 3 or 4 days,

$$P_D(d) = \begin{cases} 1/3 & d = 2, 3, 4, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Given a trip lasts d days, the weight change is equally likely to be any value between $-d$ and d pounds. Thus,

$$P_{W|D}(w|d) = \begin{cases} 1/(2d+1) & w = -d, -d+1, \ldots, d, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

The joint PMF is simply

$$\begin{aligned} P_{D,W}(d, w) &= P_{W|D}(w|d)\, P_D(d) \\ &= \begin{cases} 1/(6d+3) & d = 2, 3, 4; w = -d, \ldots, d, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{3}$$

Problem 7.4.6 Solution

$$f_{X,Y}(x, y) = \begin{cases} (x+y) & 0 \le x, y \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(a) The conditional PDF $f_{X|Y}(x|y)$ is defined for all y such that $0 \le y \le 1$. For $0 \le y \le 1$,

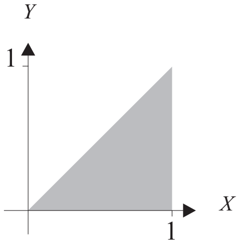
$$f_{X|Y}(x) = \frac{f_{X,Y}(x, y)}{f_Y(y)} = \frac{(x+y)}{\int_0^1 (x+y)\, dx} = \begin{cases} \frac{(x+y)}{y+1/2} & 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

(b) The conditional PDF $f_{Y|X}(y|x)$ is defined for all values of x in the interval $[0, 1]$. For $0 \le x \le 1$,

$$f_{Y|X}(y) = \frac{f_{X,Y}(x, y)}{f_X(x)} = \frac{(x+y)}{\int_0^1 (x+y)\, dy} = \begin{cases} \frac{(x+y)}{x+1/2} & 0 \le y \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Problem 7.5.1 Solution

Random variables X and Y have joint PDF



$$f_{X,Y}(x,y) = \begin{cases} 2 & 0 \le y \le x \le 1 \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

For $0 \le y \le 1$,

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx = \int_y^1 2\, dx = 2(1-y). \tag{2}$$

Also, for $y < 0$ or $y > 1$, $f_Y(y) = 0$. The complete expression for the marginal PDF is

$$f_Y(y) = \begin{cases} 2(1-y) & 0 \le y \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

By Theorem 7.10, the conditional PDF of X given Y is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \frac{1}{1-y} & y \le x \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

That is, since $Y \le X \le 1$, X is uniform over $[y,1]$ when $Y = y$. The conditional expectation of X given $Y = y$ can be calculated as

$$\mathrm{E}[X|Y=y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\, dx \tag{5}$$

$$= \int_y^1 \frac{x}{1-y}\, dx = \left. \frac{x^2}{2(1-y)} \right|_y^1 = \frac{1+y}{2}. \tag{6}$$

In fact, since we know that the conditional PDF of X is uniform over $[y,1]$ when $Y = y$, it wasn't really necessary to perform the calculation.

Problem 7.5.3 Solution

(a) First we observe that A takes on the values $S_A = \{-1, 1\}$ while B takes on values from $S_B = \{0, 1\}$. To construct a table describing $P_{A,B}(a, b)$ we build a table for all possible values of pairs (A, B). The general form of the entries is

$$\begin{array}{c|cc} P_{A,B}(a,b) & b=0 & b=1 \\ \hline a=-1 & P_{B|A}(0|-1)\,P_A(-1) & P_{B|A}(1|-1)\,P_A(-1) \\ a=1 & P_{B|A}(0|1)\,P_A(1) & P_{B|A}(1|1)\,P_A(1) \end{array} \tag{1}$$

Now we fill in the entries using the conditional PMFs $P_{B|A}(b|a)$ and the marginal PMF $P_A(a)$. This yields

$$\begin{array}{c|cc} P_{A,B}(a,b) & b=0 & b=1 \\ \hline a=-1 & (1/3)(1/3) & (2/3)(1/3) \\ a=1 & (1/2)(2/3) & (1/2)(2/3) \end{array}, \tag{2}$$

which simplifies to

$$\begin{array}{c|cc} P_{A,B}(a,b) & b=0 & b=1 \\ \hline a=-1 & 1/9 & 2/9 \\ a=1 & 1/3 & 1/3 \end{array}. \tag{3}$$

(b) Since $P_A(1) = P_{A,B}(1, 0) + P_{A,B}(1, 1) = 2/3$,

$$P_{B|A}(b|1) = \frac{P_{A,B}(1,b)}{P_A(1)} = \begin{cases} 1/2 & b = 0, 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

If $A = 1$, the conditional expectation of B is

$$\mathrm{E}[B|A=1] = \sum_{b=0}^{1} b P_{B|A}(b|1) = P_{B|A}(1|1) = 1/2. \tag{5}$$

(c) Before finding the conditional PMF $P_{A|B}(a|1)$, we first sum the columns of the joint PMF table to find

$$P_B(b) = \begin{cases} 4/9 & b = 0, \\ 5/9 & b = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

The conditional PMF of A given $B = 1$ is

$$P_{A|B}(a|1) = \frac{P_{A,B}(a,1)}{P_B(1)} = \begin{cases} 2/5 & a = -1, \\ 3/5 & a = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

(d) Now that we have the conditional PMF $P_{A|B}(a|1)$, calculating conditional expectations is easy.

$$\mathrm{E}\left[A|B=1\right] = \sum_{a=-1,1} aP_{A|B}(a|1) = -1(2/5) + (3/5) = 1/5, \tag{8}$$

$$\mathrm{E}\left[A^2|B=1\right] = \sum_{a=-1,1} a^2 P_{A|B}(a|1) = 2/5 + 3/5 = 1. \tag{9}$$

The conditional variance is then

$$\begin{aligned} \mathrm{Var}[A|B=1] &= \mathrm{E}\left[A^2|B=1\right] - (\mathrm{E}\left[A|B=1\right])^2 \\ &= 1 - (1/5)^2 = 24/25. \end{aligned} \tag{10}$$

(e) To calculate the covariance, we need

$$\mathrm{E}\left[A\right] = \sum_{a=-1,1} aP_A(a) = -1(1/3) + 1(2/3) = 1/3, \tag{11}$$

$$\mathrm{E}\left[B\right] = \sum_{b=0}^{1} bP_B(b) = 0(4/9) + 1(5/9) = 5/9, \tag{12}$$

$$\begin{aligned} \mathrm{E}\left[AB\right] &= \sum_{a=-1,1} \sum_{b=0}^{1} abP_{A,B}(a,b) \\ &= -1(0)(1/9) + -1(1)(2/9) + 1(0)(1/3) + 1(1)(1/3) \\ &= 1/9. \end{aligned} \tag{13}$$

The covariance is just

$$\begin{aligned} \operatorname{Cov}[A, B] &= \operatorname{E}[AB] - \operatorname{E}[A]\operatorname{E}[B] \\ &= 1/9 - (1/3)(5/9) = -2/27. \end{aligned} \tag{14}$$