Problem 6.4.1 Solution

Since $0 \leq X \leq 1$, and $0 \leq Y \leq 1$, we have $0 \leq V \leq 1$. This implies $F_V(v) = 0$ for

$v < 0$ and $F_V(v) = 1$ for $v \geq 1$. For $0 \leq v \leq 1$,

$$\begin{aligned} F_V(v) = \mathrm{P}[\max(X,Y) \leq v] &= \mathrm{P}[X \leq v, Y \leq v] \\ &= \int_0^v \int_0^v 6xy^2\,dx\,dy \\ &= \left(\int_0^v 2x\,dx\right)\left(\int_0^v 3y^2\,dy\right) \\ &= (v^2)(v^3) = v^5. \end{aligned} \tag{1}$$

The CDF and (by taking the derivative) PDF of V are

$$F_V(v) = \begin{cases} 0 & v < 0, \\ v^5 & 0 \leq v \leq 1, \\ 1 & v > 1, \end{cases} \qquad f_V(v) = \begin{cases} 5v^4 & 0 \leq v \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Problem 6.4.2 Solution

Since $0 \leq X \leq 1$, and $0 \leq Y \leq 1$, we have $0 \leq W \leq 1$. This implies $F_W(w) = 0$ for $w < 0$ and $F_W(w) = 1$ for $w \geq 1$. For $0 \leq w \leq 1$,

$$\begin{aligned} F_W(w) &= \mathrm{P}[\min(X,Y) \leq w] \\ &= 1 - \mathrm{P}[\min(X,Y) \geq w] = 1 - \mathrm{P}[X \geq w, Y \geq w]. \end{aligned} \tag{1}$$

Now we calculate

$$\begin{aligned} \mathrm{P}[X \geq w, Y \geq w] &= \int_w^1 \int_w^1 6xy^2\,dx\,dy \\ &= \left(\int_w^1 2x\,dx\right)\left(\int_w^1 3y^2\,dy\right) \\ &= (1-w^2)(1-w^3) = 1 - w^2 - w^3 + w^5. \end{aligned} \tag{2}$$

The complete expression for the CDF of W is

$$\begin{aligned} F_W(w) &= 1 - \mathrm{P}[X \geq w, Y \geq w] \\ &= \begin{cases} 0 & w < 0, \\ w^2 + w^3 - w^5 & 0 \leq w \leq 1, \\ 1 & w > 1. \end{cases} \end{aligned} \tag{3}$$

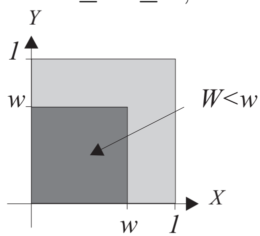
Taking the derivative of the CDF, we obtain the PDF

$$f_W(w) = \begin{cases} 2w + 3w^2 - 5w^4 & 0 \le w \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Problem 6.4.4 Solution

(a) The minimum value of W is $W = 0$, which occurs when $X = 0$ and $Y = 0$. The maximum value of W is $W = 1$, which occurs when $X = 1$ or $Y = 1$. The range of W is $S_W = \{w|0 \le w \le 1\}$.

(b) For $0 \le w \le 1$, the CDF of W is



$$\begin{aligned} F_W(w) &= P[\max(X, Y) \le w] \\ &= P[X \le w, Y \le w] \\ &= \int_0^w \int_0^w f_{X,Y}(x, y)\, dy\, dx. \end{aligned} \tag{1}$$

Substituting $f_{X,Y}(x, y) = x + y$ yields

$$\begin{aligned} F_W(w) &= \int_0^w \int_0^w (x + y)\, dy\, dx \\ &= \int_0^w \left(xy + \frac{y^2}{2}\bigg|_{y=0}^{y=w} \right) dx = \int_0^w (wx + w^2/2)\, dx = w^3. \end{aligned} \tag{2}$$

The complete expression for the CDF is

$$F_W(w) = \begin{cases} 0 & w < 0, \\ w^3 & 0 \le w \le 1, \\ 1 & \text{otherwise.} \end{cases} \tag{3}$$

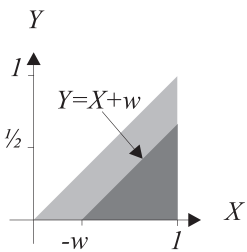
The PDF of W is found by differentiating the CDF.

$$f_W(w) = \frac{dF_W(w)}{dw} = \begin{cases} 3w^2 & 0 \le w \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Problem 6.4.5 Solution

(a) Since the joint PDF $f_{X,Y}(x,y)$ is nonzero only for $0 \le y \le x \le 1$, we observe that $W = Y - X \le 0$ since $Y \le X$. In addition, the most negative value of W occurs when $Y = 0$ and $X = 1$ and $W = -1$. Hence the range of W is $S_W = \{w| -1 \le w \le 0\}$.

(b) For $w < -1$, $F_W(w) = 0$. For $w > 0$, $F_W(w) = 1$. For $-1 \le w \le 0$, the CDF of W is



$$\begin{aligned} F_W(w) &= \mathrm{P}[Y - X \le w] \\ &= \int_{-w}^{1} \int_{0}^{x+w} 6y \, dy \, dx \\ &= \int_{-w}^{1} 3(x+w)^2 \, dx \\ &= (x+w)^3 \Big|_{-w}^{1} = (1+w)^3. \end{aligned} \tag{1}$$

Therefore, the complete CDF of W is

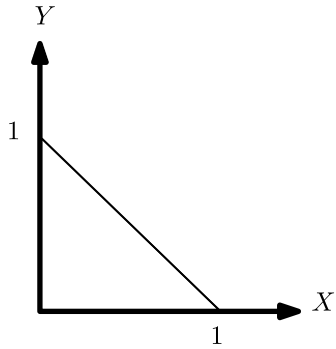
$$F_W(w) = \begin{cases} 0 & w < -1, \\ (1+w)^3 & -1 \le w \le 0, \\ 1 & w > 0. \end{cases} \tag{2}$$

By taking the derivative of $f_W(w)$ with respect to w, we obtain the PDF

$$f_W(w) = \begin{cases} 3(w+1)^2 & -1 \le w \le 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Problem 6.5.2 Solution

The key to the solution is to draw the triangular region where the PDF is nonzero:



For the PDF of $W = X + Y$, we could use the usual procedure to derive the CDF of W and take a derivative, but it is much easier to use Theorem 6.4 to write

$$f_W(w) = \int_{-\infty}^{\infty} f_{X,Y}(x, w - x)\ dx. \tag{1}$$

For $0 \leq w \leq 1$,

$$f_W(w) = \int_0^w 2\, dx = 2w. \tag{2}$$

For $w < 0$ or $w > 1$, $f_W(w) = 0$ since $0 \leq W \leq 1$. The complete expression is

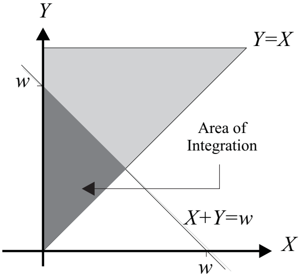
$$f_W(w) = \begin{cases} 2w & 0 \leq w \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Problem 6.5.3 Solution

The joint PDF of X and Y is

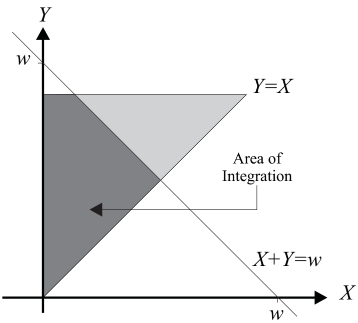
$$f_{X,Y}(x, y) = \begin{cases} 2 & 0 \leq x \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

We wish to find the PDF of W where $W = X + Y$. First we find the CDF of W, $F_W(w)$, but we must realize that the CDF will require different integrations for different values of w.



For values of $0 \leq w \leq 1$ we look to integrate the shaded area in the figure to the right.

$$F_W(w) = \int_0^{\frac{w}{2}} \int_x^{w-x} 2\, dy\, dx = \frac{w^2}{2}. \tag{2}$$



For values of w in the region $1 \leq w \leq 2$ we look to integrate over the shaded region in the graph to the right. From the graph we see that we can integrate with respect to x first, ranging y from 0 to $w/2$, thereby covering the lower right triangle of the shaded region and leaving the upper trapezoid, which is accounted for in the second term of the following expression:

$$\begin{aligned} F_W(w) &= \int_0^{\frac{w}{2}} \int_0^{y} 2\, dx\, dy + \int_{\frac{w}{2}}^{1} \int_0^{w-y} 2\, dx\, dy \\ &= 2w - 1 - \frac{w^2}{2}. \end{aligned} \tag{3}$$

Putting all the parts together gives the CDF

$$F_W(w) = \begin{cases} 0 & w < 0, \\ \frac{w^2}{2} & 0 \leq w \leq 1, \\ 2w - 1 - \frac{w^2}{2} & 1 \leq w \leq 2, \\ 1 & w > 2, \end{cases} \tag{4}$$

and (by taking the derivative) the PDF

$$f_W(w) = \begin{cases} w & 0 \leq w \leq 1, \\ 2 - w & 1 \leq w \leq 2, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Problem 6.5.4 Solution

The joint PDF of X and Y is

$$f_{X,Y}(x,y) = \begin{cases} 1 & 0 \le x, y \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Proceeding as in Problem 6.5.3, we must first find $F_W(w)$ by integrating over the square defined by $0 \le x, y \le 1$. Again we are forced to find $F_W(w)$ in parts as we did in Problem 6.5.3 resulting in the following integrals for their appropriate regions. For $0 \le w \le 1$,

$$F_W(w) = \int_0^w \int_0^{w-x} dx\, dy = w^2/2. \tag{2}$$

For $1 \le w \le 2$,

$$F_W(w) = \int_0^{w-1} \int_0^1 dx\, dy + \int_{w-1}^1 \int_0^{w-y} dx\, dy = 2w - 1 - w^2/2. \tag{3}$$

The complete CDF is

$$F_W(w) = \begin{cases} 0 & w < 0, \\ w^2/2 & 0 \le w \le 1, \\ 2w - 1 - w^2/2 & 1 \le w \le 2, \\ 1 & \text{otherwise.} \end{cases} \tag{4}$$

The corresponding PDF, $f_W(w) = dF_W(w)/dw$, is

$$f_W(w) = \begin{cases} w & 0 \le w \le 1, \\ 2 - w & 1 \le w \le 2, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Problem 6.5.5 Solution

By using Theorem 6.9, we can find the PDF of $W = X + Y$ by convolving the two exponential distributions. For $\mu \neq \lambda$,

$$\begin{aligned} f_W(w) &= \int_{-\infty}^{\infty} f_X(x)\, f_Y(w-x)\ dx \\ &= \int_0^w \lambda e^{-\lambda x} \mu e^{-\mu(w-x)}\, dx \\ &= \lambda\mu e^{-\mu w} \int_0^w e^{-(\lambda-\mu)x}\, dx \\ &= \begin{cases} \frac{\lambda\mu}{\lambda-\mu}\left(e^{-\mu w} - e^{-\lambda w}\right) & w \geq 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{1}$$

When $\mu = \lambda$, the previous derivation is invalid because of the denominator term $\lambda - \mu$. For $\mu = \lambda$, we have

$$\begin{aligned} f_W(w) &= \int_{-\infty}^{\infty} f_X(x)\, f_Y(w-x)\ dx \\ &= \int_0^w \lambda e^{-\lambda x} \lambda e^{-\lambda(w-x)}\, dx \\ &= \lambda^2 e^{-\lambda w} \int_0^w dx \\ &= \begin{cases} \lambda^2 w e^{-\lambda w} & w \geq 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{2}$$

Note that when $\mu = \lambda$, W is the sum of two iid exponential random variables and has a second order Erlang PDF.