# Probability and Stochastic Processes:
## A Friendly Introduction for Electrical and Computer Engineers
## Edition 3
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**Yates and Goodman 3e Solution Set:** 5.8.1, 5.8.3, 5.8.6, and 5.8.7

## Problem 5.8.1 Solution

Independence of $X$ and $Z$ implies

$$\text{Var}[Y] = \text{Var}[X] + \text{Var}[Z] = 1^2 + 4^2 = 17. \tag{1}$$

Since $\text{E}[X] = \text{E}[Y] = 0$, the covariance of $X$ and $Y$ is

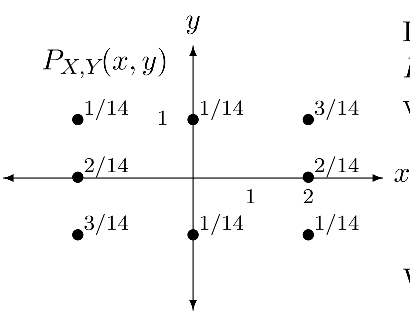
$$\text{Cov}\left[X, Y\right] = \text{E}\left[XY\right] = \text{E}\left[X(X+Z)\right] = \text{E}\left[X^2\right] + \text{E}\left[XZ\right]. \tag{2}$$

Since $X$ and $Z$ are independent, $\text{E}[XZ] = \text{E}[X]\,\text{E}[Z] = 0$. Since $\text{E}[X] = 0$, $\text{E}[X^2] = \text{Var}[X] = 1$. Thus $\text{Cov}[X, Y] = 1$. Finally, the correlation coefficient is

$$\rho_{X,Y} = \frac{\text{Cov}\left[X, Y\right]}{\sqrt{\text{Var}[X]}\sqrt{\text{Var}[Y]}} = \frac{1}{\sqrt{17}} = 0.243. \tag{3}$$

Since $\rho_{X,Y} \neq 0$, we conclude that $X$ and $Y$ are dependent.

## Problem 5.8.3 Solution




In Problem 5.2.1, we found the joint PMF $P_{X,Y}(x, y)$ shown here. The expected values and variances were found to be

$$\text{E}\left[X\right] = 0, \qquad \text{Var}[X] = 24/7, \tag{1}$$

$$\text{E}\left[Y\right] = 0, \qquad \text{Var}[Y] = 5/7. \tag{2}$$

We need these results to solve this problem.

(a) Random variable $W = 2^{XY}$ has expected value

$$\begin{aligned}
\mathrm{E}\left[2^{XY}\right] &= \sum_{x=-2,0,2} \sum_{y=-1,0,1} 2^{xy} P_{X,Y}(x,y) \\
&= 2^{-2(-1)} \frac{3}{14} + 2^{-2(0)} \frac{2}{14} + 2^{-2(1)} \frac{1}{14} + 2^{0(-1)} \frac{1}{14} + 2^{0(1)} \frac{1}{14} \\
&\quad + 2^{2(-1)} \frac{1}{14} + 2^{2(0)} \frac{2}{14} + 2^{2(1)} \frac{3}{14} \\
&= 61/28.
\end{aligned} \tag{3}$$

(b) The correlation of $X$ and $Y$ is

$$\begin{aligned}
r_{X,Y} &= \sum_{x=-2,0,2} \sum_{y=-1,0,1} xy P_{X,Y}(x,y) \\
&= \frac{-2(-1)(3)}{14} + \frac{-2(0)(2)}{14} + \frac{-2(1)(1)}{14} \\
&\qquad + \frac{2(-1)(1)}{14} + \frac{2(0)(2)}{14} + \frac{2(1)(3)}{14} \\
&= 4/7.
\end{aligned} \tag{4}$$

(c) The covariance of $X$ and $Y$ is

$$\mathrm{Cov}[X,Y] = \mathrm{E}[XY] - \mathrm{E}[X]\,\mathrm{E}[Y] = 4/7. \tag{5}$$

(d) The correlation coefficient is

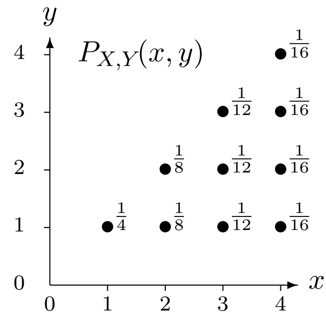
$$\rho_{X,Y} = \frac{\mathrm{Cov}[X,Y]}{\sqrt{\mathrm{Var}[X]\,\mathrm{Var}[Y]}} = \frac{2}{\sqrt{30}}. \tag{6}$$

(e) By Theorem 5.16,

$$\begin{aligned}
\mathrm{Var}[X+Y] &= \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\,\mathrm{Cov}[X,Y] \\
&= \frac{24}{7} + \frac{5}{7} + 2\frac{4}{7} = \frac{37}{7}.
\end{aligned} \tag{7}$$

## Problem 5.8.6 Solution

From the joint PMF




we can find the marginal PMF for $X$ or $Y$ by summing over the columns or rows of the joint PMF.

$$P_Y(y) = \begin{cases} 25/48 & y=1, \\ 13/48 & y=2, \\ 7/48 & y=3, \\ 3/48 & y=4, \\ 0 & \text{otherwise,} \end{cases} \qquad P_X(x) = \begin{cases} 1/4 & x=1,2,3,4, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(a) The expected values are

$$\mathrm{E}[Y] = \sum_{y=1}^{4} y P_Y(y) = 1\frac{25}{48} + 2\frac{13}{48} + 3\frac{7}{48} + 4\frac{3}{48} = 7/4, \tag{2}$$

$$\mathrm{E}[X] = \sum_{x=1}^{4} x P_X(x) = \frac{1}{4}(1+2+3+4) = 5/2. \tag{3}$$

(b) To find the variances, we first find the second moments.

$$\mathrm{E}\left[Y^2\right] = \sum_{y=1}^{4} y^2 P_Y(y) = 1^2\frac{25}{48} + 2^2\frac{13}{48} + 3^2\frac{7}{48} + 4^2\frac{3}{48} = 47/12, \tag{4}$$

$$\mathrm{E}\left[X^2\right] = \sum_{x=1}^{4} x^2 P_X(x) = \frac{1}{4}\left(1^2 + 2^2 + 3^2 + 4^2\right) = 15/2. \tag{5}$$

Now the variances are

$$\mathrm{Var}[Y] = \mathrm{E}\left[Y^2\right] - (\mathrm{E}\left[Y\right])^2 = 47/12 - (7/4)^2 = 41/48, \tag{6}$$

$$\mathrm{Var}[X] = \mathrm{E}\left[X^2\right] - (\mathrm{E}\left[X\right])^2 = 15/2 - (5/2)^2 = 5/4. \tag{7}$$

(c) To find the correlation, we evaluate the product $XY$ over all values of $X$ and $Y$. Specifically,

$$\begin{aligned} r_{X,Y} = \mathrm{E}\left[XY\right] &= \sum_{x=1}^{4}\sum_{y=1}^{x} xyP_{X,Y}(x,y) \\ &= \frac{1}{4}(1) + \frac{1}{8}(2+4) \\ &\qquad + \frac{1}{12}(3+6+9) + \frac{1}{16}(4+8+12+16) \\ &= 5. \end{aligned} \tag{8}$$

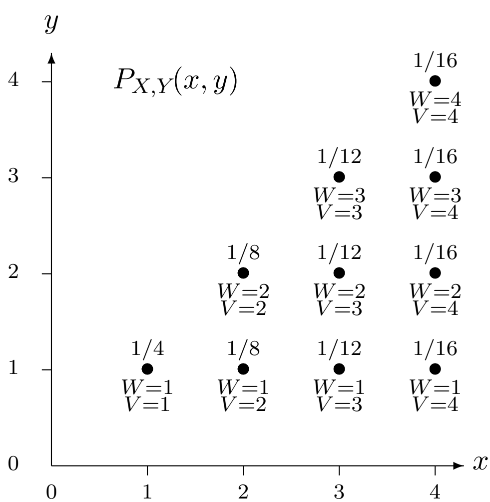
(d) The covariance of $X$ and $Y$ is

$$\mathrm{Cov}\left[X,Y\right] = \mathrm{E}\left[XY\right] - \mathrm{E}\left[X\right]\mathrm{E}\left[Y\right] = 5 - (7/4)(10/4) = 10/16. \tag{9}$$

(e) The correlation coefficient is

$$\rho_{X,Y} = \frac{\mathrm{Cov}\left[W,V\right]}{\sqrt{\mathrm{Var}[W]\,\mathrm{Var}[V]}} = \frac{10/16}{\sqrt{(41/48)(5/4)}} \approx 0.605. \tag{10}$$

## Problem 5.8.7 Solution




To solve this problem, we identify the values of $W = \min(X,Y)$ and $V = \max(X,Y)$ for each possible pair $x,y$. Here we observe that $W = Y$ and $V = X$. This is a result of the underlying experiment in that given $X = x$, each $Y \in \{1,2,\ldots,x\}$ is equally likely. Hence $Y \leq X$. This implies $\min(X,Y) = Y$ and $\max(X,Y) = X$.

Using the results from Problem 5.8.6, we have the following answers.

(a) The expected values are

$$\mathrm{E}[W] = \mathrm{E}[Y] = 7/4, \qquad \mathrm{E}[V] = \mathrm{E}[X] = 5/2. \tag{1}$$

(b) The variances are

$$\mathrm{Var}[W] = \mathrm{Var}[Y] = 41/48, \qquad \mathrm{Var}[V] = \mathrm{Var}[X] = 5/4. \tag{2}$$

(c) The correlation is

$$r_{W,V} = \mathrm{E}[WV] = \mathrm{E}[XY] = r_{X,Y} = 5. \tag{3}$$

(d) The covariance of $W$ and $V$ is

$$\mathrm{Cov}[W,V] = \mathrm{Cov}[X,Y] = 10/16. \tag{4}$$

(e) The correlation coefficient is

$$\rho_{W,V} = \rho_{X,Y} = \frac{10/16}{\sqrt{(41/48)(5/4)}}. \approx 0.605. \tag{5}$$