Probability and Stochastic Processes:
A Friendly Introduction for Electrical and Computer Engineers
Edition 3
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Yates and Goodman 3e Solution Set: 5.4.2, 5.4.3, 5.5.2, 5.5.3, 5.5.5, 5.5.8, 5.6.2, 5.6.6, 5.6.7, 5.7.2, 5.7.4, 5.7.5, and 5.7.8

Problem 5.4.2 Solution

We are given the joint PDF

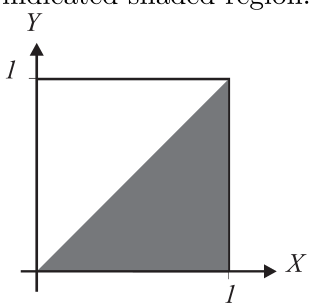
$$f_{X,Y}(x,y) = \begin{cases} cxy^2 & 0 \leq x, y \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(a) To find the constant c integrate $f_{X,Y}(x,y)$ over the all possible values of X and Y to get

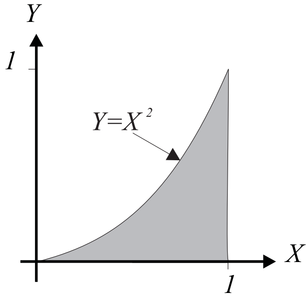
$$1 = \int_0^1 \int_0^1 cxy^2 \, dx \, dy = c/6. \tag{2}$$

Therefore $c = 6$.

(b) The probability $P[X \geq Y]$ is the integral of the joint PDF $f_{X,Y}(x,y)$ over the indicated shaded region.



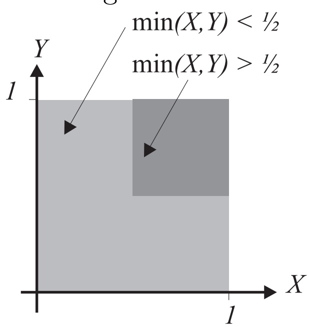
$$\begin{aligned} P[X \geq Y] &= \int_0^1 \int_0^x 6xy^2 \, dy \, dx \\ &= \int_0^1 2x^4 \, dx \\ &= 2/5. \end{aligned} \tag{3}$$



Similarly, to find $P[Y \leq X^2]$ we can integrate over the region shown in the figure.

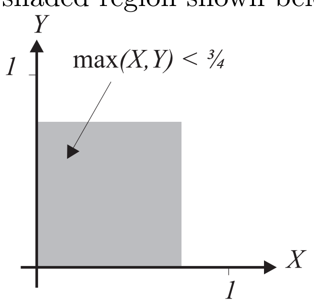
$$P\left[Y \leq X^2\right] = \int_0^1 \int_0^{x^2} 6xy^2\,dy\,dx = 1/4. \tag{4}$$

(c) Here we can choose to either integrate $f_{X,Y}(x, y)$ over the lighter shaded region, which would require the evaluation of two integrals, or we can perform one integral over the darker region by recognizing



$$\begin{aligned} P\left[\min(X,Y) \leq 1/2\right] &= 1 - P\left[\min(X,Y) > 1/2\right] \\ &= 1 - \int_{1/2}^1 \int_{1/2}^1 6xy^2\,dx\,dy \\ &= 1 - \int_{1/2}^1 \frac{9y^2}{4}\,dy \\ &= \frac{11}{32}. \end{aligned} \tag{5}$$

(d) The probability $P[\max(X, Y) \leq 3/4]$ can be found be integrating over the shaded region shown below.



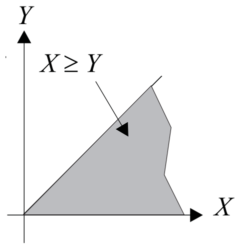
$$\begin{aligned} P\left[\max(X,Y) \leq 3/4\right] &= P\left[X \leq 3/4, Y \leq 3/4\right] \\ &= \int_0^{\frac{3}{4}} \int_0^{\frac{3}{4}} 6xy^2\,dx\,dy \\ &= \left(x^2\Big|_0^{3/4}\right)\left(y^3\Big|_0^{3/4}\right) \\ &= (3/4)^5 = 0.237. \end{aligned} \tag{6}$$

Problem 5.4.3 Solution

The joint PDF of X and Y is

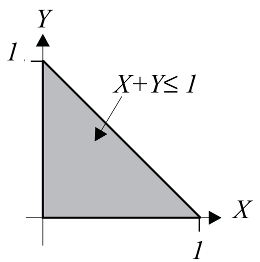
$$f_{X,Y}(x,y) = \begin{cases} 6e^{-(2x+3y)} & x \geq 0, y \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(a) The probability that $X \geq Y$ is:



$$\begin{aligned} P[X \geq Y] &= \int_0^\infty \int_0^x 6e^{-(2x+3y)}\,dy\,dx \\ &= \int_0^\infty 2e^{-2x}\left(-e^{-3y}\Big|_{y=0}^{y=x}\right)dx \\ &= \int_0^\infty [2e^{-2x} - 2e^{-5x}]\,dx = 3/5. \end{aligned} \tag{2}$$

The probability $P[X + Y \leq 1]$ is found by integrating over the region where $X + Y \leq 1$:



$$\begin{aligned} P[X + Y \leq 1] &= \int_0^1 \int_0^{1-x} 6e^{-(2x+3y)}\,dy\,dx \\ &= \int_0^1 2e^{-2x}\left[-e^{-3y}\Big|_{y=0}^{y=1-x}\right]dx \\ &= \int_0^1 2e^{-2x}\left[1 - e^{-3(1-x)}\right]dx \\ &= -e^{-2x} - 2e^{x-3}\Big|_0^1 \\ &= 1 + 2e^{-3} - 3e^{-2}. \end{aligned} \tag{3}$$

(b) The event $\{\min(X,Y) \geq 1\}$ is the same as the event $\{X \geq 1, Y \geq 1\}$. Thus,

$$P[\min(X,Y) \geq 1] = \int_1^\infty \int_1^\infty 6e^{-(2x+3y)}\,dy\,dx = e^{-(2+3)}. \tag{4}$$

(c) The event $\{\max(X,Y) \leq 1\}$ is the same as the event $\{X \leq 1, Y \leq 1\}$ so that

$$P[\max(X,Y) \leq 1] = \int_0^1 \int_0^1 6e^{-(2x+3y)}\,dy\,dx = (1 - e^{-2})(1 - e^{-3}). \tag{5}$$

Problem 5.5.2 Solution

(a) The integral of the PDF over all x, y must be unity. Thus

$$1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dx\, dy$$
$$= \int_0^1 \int_0^1 cx\, dx\, dy = c\left(\int_0^1 x\, dx\right)\left(\int_0^1 dy\right) = \frac{c}{2}. \tag{1}$$

Thus $c = 2$.

(b) The marginal PDF of X is $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy$. This integral is zero for $x < 0$ or $x > 1$. For $0 \leq x \leq 1$,

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dy = \int_0^1 2x\, dy = 2x. \tag{2}$$

Thus,

$$f_X(x) = \begin{cases} 2x & 0 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

(c) To check independence, we check whether $f_{X,Y}(x,y)$ equals $f_X(x)f_Y(y)$. Since $0 \leq Y \leq 1$, we know that $f_Y(y) = 0$ for $y < 0$ or $y > 1$. For $0 \leq y \leq 1$,

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dx = \int_0^1 2x\, dx = x^2\Big|_0^1 = 1. \tag{4}$$

Thus Y has the uniform PDF

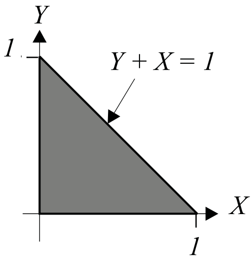
$$f_Y(y) = \begin{cases} 1 & 0 \leq y \leq 1, \\ 0 & \text{otherwise,} \end{cases}$$

and we see that $f_{X,Y}(x,y) = f_X(x)f_Y(y)$. Thus X and Y are independent.

Problem 5.5.3 Solution

X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 2 & x+y \leq 1, x, y \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$



Using the figure to the left we can find the marginal PDFs by integrating over the appropriate regions.

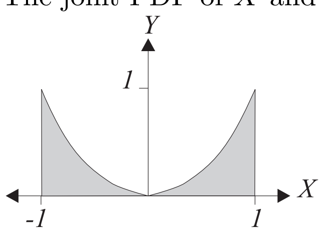
$$f_X(x) = \int_0^{1-x} 2\,dy = \begin{cases} 2(1-x) & 0 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Likewise for $f_Y(y)$:

$$f_Y(y) = \int_0^{1-y} 2\,dx = \begin{cases} 2(1-y) & 0 \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Problem 5.5.5 Solution

The joint PDF of X and Y and the region of nonzero probability are



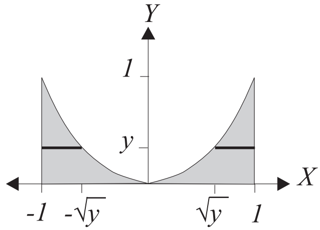
$$f_{X,Y}(x,y) = \begin{cases} 5x^2/2 & -1 \leq x \leq 1, 0 \leq y \leq x^2, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

We can find the appropriate marginal PDFs by integrating the joint PDF.

(a) The marginal PDF of X is

$$f_X(x) = \int_0^{x^2} \frac{5x^2}{2}\,dy = \begin{cases} 5x^4/2 & -1 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

(b) Note that $f_Y(y) = 0$ for $y > 1$ or $y < 0$. For $0 \le y \le 1$,



$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx \\ &= \int_{-1}^{-\sqrt{y}} \frac{5x^2}{2}\,dx + \int_{\sqrt{y}}^{1} \frac{5x^2}{2}\,dx \\ &= 5(1 - y^{3/2})/3. \end{aligned} \tag{3}$$

The complete expression for the marginal CDF of Y is

$$f_Y(y) = \begin{cases} 5(1-y^{3/2})/3 & 0 \le y \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Problem 5.5.8 Solution

The joint PDF of X and Y is

$$f_{X,Y}(x,y) = \begin{cases} (x+y)/3 & 0 \le x \le 1, 0 \le y \le 2, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(a) We first find the marginal PDFs of X and Y. For $0 \le x \le 1$,

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy \\ &= \int_0^2 \frac{x+y}{3}\,dy = \frac{xy}{3} + \frac{y^2}{6}\bigg|_{y=0}^{y=2} = \frac{2x+2}{3}. \end{aligned} \tag{2}$$

For $0 \le y \le 2$,

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx \\ &= \int_0^1 \left(\frac{x}{3} + \frac{y}{3}\right) dx = \frac{x^2}{6} + \frac{xy}{3}\bigg|_{x=0}^{x=1} = \frac{2y+1}{6}. \end{aligned} \tag{3}$$

Complete expressions for the marginal PDFs are

$$f_X(x) = \begin{cases} \frac{2x+2}{3} & 0 \le x \le 1, \\ 0 & \text{otherwise}, \end{cases} \qquad f_Y(y) = \begin{cases} \frac{2y+1}{6} & 0 \le y \le 2, \\ 0 & \text{otherwise}. \end{cases} \tag{4}$$

(b) The expected value of X is

$$\begin{aligned} \mathrm{E}[X] &= \int_{-\infty}^{\infty} x f_X(x)\, dx \\ &= \int_0^1 x \frac{2x+2}{3}\, dx = \frac{2x^3}{9} + \frac{x^2}{3}\Big|_0^1 = \frac{5}{9}. \end{aligned} \tag{5}$$

The second moment of X is

$$\begin{aligned} \mathrm{E}\left[X^2\right] &= \int_{-\infty}^{\infty} x^2 f_X(x)\, dx \\ &= \int_0^1 x^2 \frac{2x+2}{3}\, dx = \frac{x^4}{6} + \frac{2x^3}{9}\Big|_0^1 = \frac{7}{18}. \end{aligned} \tag{6}$$

The variance of X is

$$\mathrm{Var}[X] = \mathrm{E}\left[X^2\right] - (\mathrm{E}[X])^2 = 7/18 - (5/9)^2 = 13/162. \tag{7}$$

(c) The expected value of Y is

$$\begin{aligned} \mathrm{E}[Y] &= \int_{-\infty}^{\infty} y f_Y(y)\, dy \\ &= \int_0^2 y \frac{2y+1}{6}\, dy = \frac{y^2}{12} + \frac{y^3}{9}\Big|_0^2 = \frac{11}{9}. \end{aligned} \tag{8}$$

The second moment of Y is

$$\begin{aligned} \mathrm{E}\left[Y^2\right] &= \int_{-\infty}^{\infty} y^2 f_Y(y)\, dy \\ &= \int_0^2 y^2 \frac{2y+1}{6}\, dy = \frac{y^3}{18} + \frac{y^4}{12}\Big|_0^2 = \frac{16}{9}. \end{aligned} \tag{9}$$

The variance of Y is $\mathrm{Var}[Y] = \mathrm{E}[Y^2] - (\mathrm{E}[Y])^2 = 23/81$.

Problem 5.6.2 Solution

The key to this problem is understanding that "Factory Q" and "Factory R" are synonyms for $M = 60$ and $M = 180$. Similarly, "small", "medium", and "large" orders correspond to the events $B = 1$, $B = 2$ and $B = 3$.

(a) The following table given in the problem statement

	Factory Q	Factory R
small order	0.3	0.2
medium order	0.1	0.2
large order	0.1	0.1

can be translated into the following joint PMF for B and M.

$P_{B,M}(b,m)$	$m = 60$	$m = 180$
$b = 1$	0.3	0.2
$b = 2$	0.1	0.2
$b = 3$	0.1	0.1

(1)

(b) Before we find $E[B]$, it will prove helpful to find the marginal PMFs $P_B(b)$ and $P_M(m)$. These can be found from the row and column sums of the table of the joint PMF

$P_{B,M}(b,m)$	$m = 60$	$m = 180$	$P_B(b)$
$b = 1$	0.3	0.2	0.5
$b = 2$	0.1	0.2	0.3
$b = 3$	0.1	0.1	0.2
$P_M(m)$	0.5	0.5	

(2)

The expected number of boxes is

$$E[B] = \sum_b bP_B(b) = 1(0.5) + 2(0.3) + 3(0.2) = 1.7. \tag{3}$$

(c) From the marginal PMFs we calculated in the table of part (b), we can conclude that B and M are not independent. since $P_{B,M}(1,60) \neq P_B(1)P_M(m)60$.

Problem 5.6.6 Solution

X_1 and X_2 are independent random variables such that X_i has PDF

$$f_{X_i}(x)\begin{cases}\lambda_i e^{-\lambda_i x} & x \geq 0,\\ 0 & \text{otherwise.}\end{cases} \tag{1}$$

To calculate $P[X_2 < X_1]$, we use the joint PDF $f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1)f_{X_2}(x_2)$.

$$\begin{aligned} P[X_2 < X_1] &= \iint_{x_2<x_1} f_{X_1}(x_1)\, f_{X_2}(x_2)\, dx_1\, dx_2 \\ &= \int_0^\infty \lambda_2 e^{-\lambda_2 x_2} \int_{x_2}^\infty \lambda_1 e^{-\lambda_1 x_1}\, dx_1\, dx_2 \\ &= \int_0^\infty \lambda_2 e^{-\lambda_2 x_2} e^{-\lambda_1 x_2}\, dx_2 \\ &= \int_0^\infty \lambda_2 e^{-(\lambda_1+\lambda_2)x_2}\, dx_2 = \frac{\lambda_2}{\lambda_1+\lambda_2} \end{aligned} \tag{2}$$

Problem 5.6.7 Solution

(a) We find k by the requirement that the joint PDF integrate to 1. That is,

$$\begin{aligned} 1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx\, dy &= \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} (k+3x^2)\, dx\, dy \\ &= \left(\int_{-1/2}^{1/2} dy\right)\left(\int_{-1/2}^{1/2} (k+3x^2)\, dx\right) \\ &= kx + x^3\Big|_{x=-1/2}^{x=1/2} = k + 1/4 \end{aligned} \tag{1}$$

Thus k=3/4.

(b) For $-1/2 \leq x \leq 1/2$, the marginal PDF of X is

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy = \int_{-1/2}^{1/2} (k+3x^2)\, dy = k + 3x^2. \tag{2}$$

The complete expression for the PDF of X is

$$f_X(x) = \begin{cases} k + 3x^2 & -1/2 \le x \le 1/2, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

(c) For $-1/2 \le y \le 1/2$,

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_X(x)\, dx \\ &= \int_{-1/2}^{1/2} (k + 3x^2)\, dx = kx + x^3 \Big|_{x=-1/2}^{x=1/2} = k + 1/4. \end{aligned} \tag{4}$$

Since $k = 3/4$, Y is a continuous uniform $(-1/2, 1/2)$ random variable with PDF

$$f_Y(y) = \begin{cases} 1 & -1/2 \le y \le 1/2, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

(d) We need to check whether $f_{X,Y}(x,y) = f_X(x)f_Y(y)$. If you solved for k in part (a), then from (b) and (c) it is obvious that this equality holds and thus X and Y are independent. If you were not able to solve for k in part (a), testing whether $f_{X,Y}(x,y) = f_X(x)f_Y(y)$ yields the requirement $1 = k + 1/4$. With some thought, you should have gone back to check that $k = 3/4$ solves part (a). This would lead to the correct conclusion that X and Y are independent.

Problem 5.7.2 Solution

We recall that B and M have joint PMF

$P_{B,M}(b,m)$	$m = 60$	$m = 180$
$b = 1$	0.3	0.2
$b = 2$	0.1	0.2
$b = 3$	0.1	0.1

(1)

In terms of M and B, the cost (in cents) of sending a shipment is $C = BM$. The expected value of C is

$$\begin{aligned} \mathrm{E}[C] &= \sum_{b,m} bm P_{B,M}(b,m) \\ &= 1(60)(0.3) + 2(60)(0.1) + 3(60)(0.1) \\ &\quad + 1(180)(0.2) + 2(180)(0.2) + 3(180)(0.1) = 210\,\text{cents}. \end{aligned} \tag{2}$$

Problem 5.7.4 Solution

Using the following probability model

$$P_X(k) = P_Y(k) = \begin{cases} 3/4 & k = 0, \\ 1/4 & k = 20, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

We can calculate the requested moments.

$$\mathrm{E}[X] = 3/4 \cdot 0 + 1/4 \cdot 20 = 5. \tag{2}$$

$$\mathrm{Var}[X] = 3/4 \cdot (0-5)^2 + 1/4 \cdot (20-5)^2 = 75. \tag{3}$$

$$\mathrm{E}[X+Y] = \mathrm{E}[X] + \mathrm{E}[X] = 2\,\mathrm{E}[X] = 10. \tag{4}$$

Since X and Y are independent, Theorem 5.17 yields

$$\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] = 2\,\mathrm{Var}[X] = 150 \tag{5}$$

Since X and Y are independent, $P_{X,Y}(x,y) = P_X(x)P_Y(y)$ and

$$\begin{aligned} \mathrm{E}\left[XY2^{XY}\right] &= \sum_{x=0,20} \sum_{y=0,20} XY2^{XY} P_{X,Y}(x,y) \\ &= (20)(20)2^{20(20)} P_X(20)\, P_Y(20) = 2.75 \times 10^{12}. \end{aligned} \tag{6}$$

Problem 5.7.5 Solution

We start by observing that

$$\mathrm{Cov}[X,Y] = \rho\sqrt{\mathrm{Var}[X]\,\mathrm{Var}[Y]} = 1.$$

This implies

$$\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\,\mathrm{Cov}[X,Y] = 1 + 4 + 2(1) = 7.$$

Problem 5.7.8 Solution

(a) Since $\mathrm{E}[-X_2] = -\mathrm{E}[X_2]$, we can use Theorem 5.10 to write

$$\begin{aligned} \mathrm{E}[X_1 - X_2] = \mathrm{E}[X_1 + (-X_2)] &= \mathrm{E}[X_1] + \mathrm{E}[-X_2] \\ &= \mathrm{E}[X_1] - \mathrm{E}[X_2] \\ &= 0. \end{aligned} \tag{1}$$

(b) By Theorem 4.5(f), $\mathrm{Var}[-X_2] = (-1)^2 \mathrm{Var}[X_2] = \mathrm{Var}[X_2]$. Since X_1 and X_2 are independent, Theorem 5.17(a) says that

$$\begin{aligned} \mathrm{Var}[X_1 - X_2] &= \mathrm{Var}[X_1 + (-X_2)] \\ &= \mathrm{Var}[X_1] + \mathrm{Var}[-X_2] \\ &= 2\,\mathrm{Var}[X]. \end{aligned} \tag{2}$$