Probability and Stochastic Processes:
A Friendly Introduction for Electrical and Computer Engineers
Edition 3
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Yates and Goodman 3e Solution Set: 5.1.1, 5.2.2, 5.2.3, 5.2.4, 5.3.2, 5.3.3, and 5.3.4

Problem 5.1.1 Solution

(a) The probability $P[X \leq 2, Y \leq 3]$ can be found be evaluating the joint CDF $F_{X,Y}(x,y)$ at $x = 2$ and $y = 3$. This yields

$$P[X \leq 2, Y \leq 3] = F_{X,Y}(2,3) = (1 - e^{-2})(1 - e^{-3}) \tag{1}$$

(b) To find the marginal CDF of X, $F_X(x)$, we simply evaluate the joint CDF at $y = \infty$.

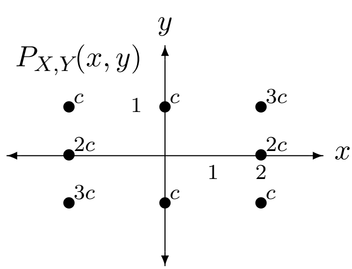
$$F_X(x) = F_{X,Y}(x,\infty) = \begin{cases} 1 - e^{-x} & x \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

(c) Likewise for the marginal CDF of Y, we evaluate the joint CDF at $X = \infty$.

$$F_Y(y) = F_{X,Y}(\infty,y) = \begin{cases} 1 - e^{-y} & y \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Problem 5.2.2 Solution

On the X, Y plane, the joint PMF is



(a) To find c, we sum the PMF over all possible values of X and Y. We choose c so the sum equals one.

$$\sum_x \sum_y P_{X,Y}(x,y) = \sum_{x=-2,0,2} \sum_{y=-1,0,1} c\,|x+y| = 6c + 2c + 6c = 14c. \tag{1}$$

Thus $c = 1/14$.

(b)

$$\begin{aligned} P[Y < X] &= P_{X,Y}(0,-1) + P_{X,Y}(2,-1) + P_{X,Y}(2,0) + P_{X,Y}(2,1) \\ &= c + c + 2c + 3c = 7c = 1/2. \end{aligned} \tag{2}$$

(c)

$$\begin{aligned} P[Y > X] &= P_{X,Y}(-2,-1) + P_{X,Y}(-2,0) + P_{X,Y}(-2,1) + P_{X,Y}(0,1) \\ &= 3c + 2c + c + c = 7c = 1/2. \end{aligned} \tag{3}$$

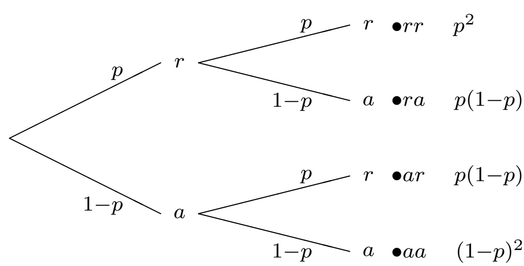
(d) From the sketch of $P_{X,Y}(x,y)$ given above, $P[X = Y] = 0$.

(e)

$$\begin{aligned} P[X < 1] = P_{X,Y}(-2,-1) &+ P_{X,Y}(-2,0) + P_{X,Y}(-2,1) \\ &+ P_{X,Y}(0,-1) + P_{X,Y}(0,1) \\ &= 8c = 8/14. \end{aligned} \tag{4}$$

Problem 5.2.3 Solution

Let r (reject) and a (accept) denote the result of each test. There are four possible outcomes: rr, ra, ar, aa. The sample tree is



Now we construct a table that maps the sample outcomes to values of X and Y.

outcome	$P[\cdot]$	X	Y
rr	p^2	1	1
ra	$p(1-p)$	1	0
ar	$p(1-p)$	0	1
aa	$(1-p)^2$	0	0

(1)

This table is esentially the joint PMF $P_{X,Y}(x, y)$.

$$P_{X,Y}(x,y) = \begin{cases} p^2 & x=1, y=1, \\ p(1-p) & x=0, y=1, \\ p(1-p) & x=1, y=0, \\ (1-p)^2 & x=0, y=0, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Problem 5.2.4 Solution

The sample space is the set $S = \{hh, ht, th, tt\}$ and each sample point has probability 1/4. Each sample outcome specifies the values of X and Y as given in the following table

outcome	X	Y
hh	0	1
ht	1	0
th	1	1
tt	2	0

(1)

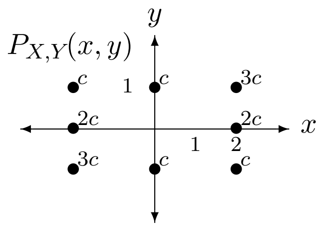
The joint PMF can represented by the table

$P_{X,Y}(x,y)$	$y=0$	$y=1$
$x=0$	0	1/4
$x=1$	1/4	1/4
$x=2$	1/4	0

(2)

Problem 5.3.2 Solution

On the X, Y plane, the joint PMF is



The PMF sums to one when $c = 1/14$.

(a) The marginal PMFs of X and Y are

$$P_X(x) = \sum_{y=-1,0,1} P_{X,Y}(x, y) = \begin{cases} 6/14 & x = -2, 2, \\ 2/14 & x = 0, \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

$$P_Y(y) = \sum_{x=-2,0,2} P_{X,Y}(x, y) = \begin{cases} 5/14 & y = -1, 1, \\ 4/14 & y = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

(b) The expected values of X and Y are

$$\mathrm{E}[X] = \sum_{x=-2,0,2} x P_X(x) = -2(6/14) + 2(6/14) = 0, \tag{3}$$

$$\mathrm{E}[Y] = \sum_{y=-1,0,1} y P_Y(y) = -1(5/14) + 1(5/14) = 0. \tag{4}$$

(c) Since X and Y both have zero mean, the variances are

$$\begin{aligned}\operatorname{Var}[X] = \mathrm{E}\left[X^2\right] &= \sum_{x=-2,0,2} x^2 P_X(x) \\ &= (-2)^2(6/14) + 2^2(6/14) = 24/7, \qquad (5)\\ \operatorname{Var}[Y] = \mathrm{E}\left[Y^2\right] &= \sum_{y=-1,0,1} y^2 P_Y(y) \\ &= (-1)^2(5/14) + 1^2(5/14) = 5/7. \qquad (6)\end{aligned}$$

The standard deviations are $\sigma_X = \sqrt{24/7}$ and $\sigma_Y = \sqrt{5/7}$.

Problem 5.3.3 Solution

We recognize that the given joint PMF is written as the product of two marginal PMFs $P_N(n)$ and $P_K(k)$ where

$$P_N(n) = \sum_{k=0}^{100} P_{N,K}(n,k) = \begin{cases} \frac{100^n e^{-100}}{n!} & n = 0, 1, \ldots, \\ 0 & \text{otherwise}, \end{cases} \qquad (1)$$

$$P_K(k) = \sum_{n=0}^{\infty} P_{N,K}(n,k) = \begin{cases} \binom{100}{k} p^k (1-p)^{100-k} & k = 0, 1, \ldots, 100, \\ 0 & \text{otherwise}. \end{cases} \qquad (2)$$

Problem 5.3.4 Solution

For integers $0 \leq x \leq 5$, the marginal PMF of X is

$$P_X(x) = \sum_y P_{X,Y}(x,y) = \sum_{y=0}^{x} (1/21) = \frac{x+1}{21}. \qquad (1)$$

Similarly, for integers $0 \leq y \leq 5$, the marginal PMF of Y is

$$P_Y(y) = \sum_x P_{X,Y}(x,y) = \sum_{x=y}^{5} (1/21) = \frac{6-y}{21}. \qquad (2)$$

The complete expressions for the marginal PMFs are

$$P_X(x) = \begin{cases} (x+1)/21 & x = 0, \dots, 5, \\ 0 & \text{otherwise}, \end{cases} \tag{3}$$

$$P_Y(y) = \begin{cases} (6-y)/21 & y = 0, \dots, 5, \\ 0 & \text{otherwise}. \end{cases} \tag{4}$$

The expected values are

$$\mathrm{E}[X] = \sum_{x=0}^{5} x \frac{x+1}{21} = \frac{70}{21} = \frac{10}{3}, \tag{5}$$

$$\mathrm{E}[Y] = \sum_{y=0}^{5} y \frac{6-y}{21} = \frac{35}{21} = \frac{5}{3}. \tag{6}$$