Probability and Stochastic Processes:
A Friendly Introduction for Electrical and Computer Engineers
Edition 3
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Yates and Goodman 3e Solution Set: 4.5.4, 4.5.5, 4.5.6, 4.5.7, 4.5.10, 4.5.12, 4.6.3, 4.6.4, 4.6.6, and 4.6.10

Problem 4.5.4 Solution

From Appendix A, we observe that an exponential PDF Y with parameter $\lambda > 0$ has PDF

$$f_Y(y) = \begin{cases} \lambda e^{-\lambda y} & y \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

In addition, the mean and variance of Y are

$$\mathrm{E}[Y] = \frac{1}{\lambda}, \qquad \mathrm{Var}[Y] = \frac{1}{\lambda^2}. \tag{2}$$

(a) Since $\mathrm{Var}[Y] = 25$, we must have $\lambda = 1/5$.

(b) The expected value of Y is $\mathrm{E}[Y] = 1/\lambda = 5$.

(c)

$$\mathrm{P}[Y > 5] = \int_5^\infty f_Y(y)\, dy = -e^{-y/5}\Big|_5^\infty = e^{-1}. \tag{3}$$

Problem 4.5.5 Solution

An exponential (λ) random variable has PDF

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0, \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

and has expected value $\mathrm{E}[Y] = 1/\lambda$. Although λ was not specified in the problem, we can still solve for the probabilities:

(a) $\mathrm{P}[Y \geq \mathrm{E}[Y]] = \int_{1/\lambda}^{\infty} \lambda e^{-\lambda x}\, dx = -e^{-\lambda x}\Big|_{1/\lambda}^{\infty} = e^{-1}.$

(b) $\mathrm{P}[Y \geq 2\,\mathrm{E}[Y]] = \int_{2/\lambda}^{\infty} \lambda e^{-\lambda x}\, dx = -e^{-\lambda x}\Big|_{2/\lambda}^{\infty} = e^{-2}.$

Problem 4.5.6 Solution

From Appendix A, an Erlang random variable X with parameters $\lambda > 0$ and n has PDF

$$f_X(x) = \begin{cases} \lambda^n x^{n-1} e^{-\lambda x}/(n-1)! & x \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

In addition, the mean and variance of X are

$$\mathrm{E}[X] = \frac{n}{\lambda}, \qquad \mathrm{Var}[X] = \frac{n}{\lambda^2}. \tag{2}$$

(a) Since $\lambda = 1/3$ and $\mathrm{E}[X] = n/\lambda = 15$, we must have $n = 5$.

(b) Substituting the parameters $n = 5$ and $\lambda = 1/3$ into the given PDF, we obtain

$$f_X(x) = \begin{cases} (1/3)^5 x^4 e^{-x/3}/24 & x \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

(c) From above, we know that $\mathrm{Var}[X] = n/\lambda^2 = 45$.

Problem 4.5.7 Solution

Since Y is an Erlang random variable with parameters $\lambda = 2$ and $n = 2$, we find in Appendix A that

$$f_Y(y) = \begin{cases} 4ye^{-2y} & y \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(a) Appendix A tells us that $\mathrm{E}[Y] = n/\lambda = 1$.

(b) Appendix A also tells us that $\text{Var}[Y] = n/\lambda^2 = 1/2$.

(c) The probability that $1/2 \le Y < 3/2$ is

$$P[1/2 \le Y < 3/2] = \int_{1/2}^{3/2} f_Y(y)\, dy = \int_{1/2}^{3/2} 4ye^{-2y}\, dy. \tag{2}$$

This integral is easily completed using the integration by parts formula $\int u\, dv = uv - \int v\, du$ with

$$\begin{aligned} u &= 2y, & dv &= 2e^{-2y}, \\ du &= 2dy, & v &= -e^{-2y}. \end{aligned}$$

Making these substitutions, we obtain

$$\begin{aligned} P[1/2 \le Y < 3/2] &= -2ye^{-2y}\Big|_{1/2}^{3/2} + \int_{1/2}^{3/2} 2e^{-2y}\, dy \\ &= 2e^{-1} - 4e^{-3} = 0.537. \end{aligned} \tag{3}$$

Problem 4.5.10 Solution

(a) The PDF of a continuous uniform $(-5, 5)$ random variable is

$$f_X(x) = \begin{cases} 1/10 & -5 \le x \le 5, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(b) For $x < -5$, $F_X(x) = 0$. For $x \ge 5$, $F_X(x) = 1$. For $-5 \le x \le 5$, the CDF is

$$F_X(x) = \int_{-5}^{x} f_X(\tau)\, d\tau = \frac{x+5}{10}. \tag{2}$$

The complete expression for the CDF of X is

$$F_X(x) = \begin{cases} 0 & x < -5, \\ (x+5)/10 & 5 \le x \le 5, \\ 1 & x > 5. \end{cases} \tag{3}$$

(c) The expected value of X is

$$\int_{-5}^{5} \frac{x}{10}\,dx = \left.\frac{x^2}{20}\right|_{-5}^{5} = 0. \tag{4}$$

Another way to obtain this answer is to use Theorem 4.6 which says the expected value of X is $\mathrm{E}[X] = (5 + -5)/2 = 0$.

(d) The fifth moment of X is

$$\int_{-5}^{5} \frac{x^5}{10}\,dx = \left.\frac{x^6}{60}\right|_{-5}^{5} = 0. \tag{5}$$

(e) The expected value of e^X is

$$\int_{-5}^{5} \frac{e^x}{10}\,dx = \left.\frac{e^x}{10}\right|_{-5}^{5} = \frac{e^5 - e^{-5}}{10} = 14.84. \tag{6}$$

Problem 4.5.12 Solution

We know that X has a uniform PDF over $[a, b)$ and has mean $\mu_X = 7$ and variance $\mathrm{Var}[X] = 3$. All that is left to do is determine the values of the constants a and b, to complete the model of the uniform PDF.

$$\mathrm{E}[X] = \frac{a+b}{2} = 7, \qquad \mathrm{Var}[X] = \frac{(b-a)^2}{12} = 3. \tag{1}$$

Since we assume $b > a$, this implies

$$a + b = 14, \qquad b - a = 6. \tag{2}$$

Solving these two equations, we arrive at

$$a = 4, \qquad b = 10. \tag{3}$$

And the resulting PDF of X is,

$$f_X(x) = \begin{cases} 1/6 & 4 \le x \le 10, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Problem 4.6.3 Solution

(a)

$$P[V > 4] = 1 - P[V \leq 4] = 1 - P\left[\frac{V-0}{\sigma} \leq \frac{4-0}{\sigma}\right] \\ = 1 - \Phi(4/\sigma) \\ = 1 - \Phi(2) = 0.023. \tag{1}$$

(b)

$$P[W \leq 2] = P\left[\frac{W-2}{5} \leq \frac{2-2}{5}\right] = \Phi(0) = \frac{1}{2}. \tag{2}$$

(c)

$$\begin{aligned} P[X \leq \mu + 1] &= P[X - \mu \leq 1] \\ &= P\left[\frac{X-\mu}{\sigma} \leq \frac{1}{\sigma}\right] \\ &= \Phi(1/\sigma) = \Phi(0.5) = 0.692. \end{aligned} \tag{3}$$

(d)

$$\begin{aligned} P[Y > 65] &= 1 - P[Y \leq 65] \\ &= 1 - P\left[\frac{Y-50}{10} \leq \frac{65-50}{10}\right] \\ &= 1 - \Phi(1.5) = 1 - 0.933 = 0.067. \end{aligned} \tag{4}$$

Problem 4.6.4 Solution

In each case, we are told just enough to determine μ.

(a) Since $0.933 = \Phi(1.5)$,

$$\Phi(1.5) = P[X \leq 10] = P\left[\frac{X-\mu}{\sigma} \leq \frac{10-\mu}{\sigma}\right] = \Phi\left(\frac{10-\mu}{\sigma}\right). \tag{1}$$

It follows that

$$\frac{10-\mu}{\sigma} = 1.5,$$

or $\mu = 10 - 1.5\sigma = -5$.

(b)

$$\begin{aligned}
\mathrm{P}[Y \leq 0] &= \mathrm{P}\left[\frac{Y-\mu}{\sigma} \leq \frac{0-\mu}{\sigma}\right] \\
&= \Phi\left(\frac{-\mu}{10}\right) = 1 - \Phi\left(\frac{\mu}{10}\right) = 0.067. \qquad (2)
\end{aligned}$$

It follows that $\Phi(\mu/10) = 0.933$ and from the table we see that $\mu/10 = 1.5$ or $\mu = 15$.

(c) From the problem statement,

$$\mathrm{P}[Y \leq 10] = \mathrm{P}\left[\frac{Y-\mu}{\sigma} \leq \frac{10-\mu}{\sigma}\right] = \Phi\left(\frac{10-\mu}{\sigma}\right) = 0.977. \qquad (3)$$

From the $\Phi(\cdot)$ table,

$$\frac{10-\mu}{\sigma} = 2 \qquad \Longrightarrow \qquad \sigma = 5 - \mu/2. \qquad (4)$$

(d) Here we are not told the standard deviation σ. However, since $\mathrm{P}[Y \leq 5] = 1 - \mathrm{P}[Y > 5] = 1/2$ and Y is a Gaussian (μ, σ), we know that

$$\mathrm{P}[Y \leq 5] = \mathrm{P}\left[\frac{Y-\mu}{\sigma} \leq \frac{5-\mu}{\sigma}\right] = \Phi\left(\frac{5-\mu}{\sigma}\right) = \frac{1}{2}. \qquad (5)$$

Since $\Phi(0) = 1/2$, we see that

$$\Phi\left(\frac{5-\mu}{\sigma}\right) = \Phi(0) \qquad \Longrightarrow \qquad \mu = 5. \qquad (6)$$

Problem 4.6.6 Solution

Using σ_T to denote the (unknown) standard deviation of T, we can write

$$\begin{aligned} P[T < 66] &= P\left[\frac{T-68}{\sigma_T} < \frac{66-68}{\sigma_T}\right] \\ &= \Phi\left(\frac{-2}{\sigma_T}\right) = 1 - \Phi\left(\frac{2}{\sigma_T}\right) = 0.1587. \end{aligned} \tag{1}$$

Thus $\Phi(2/\sigma_T) = 0.8413 = \Phi(1)$. This implies $\sigma_T = 2$ and thus T has variance $\text{Var}[T] = 4$.

Problem 4.6.10 Solution

In this problem, we use Theorem 4.14 and the tables for the Φ and Q functions to answer the questions. Since $E[Y_{20}] = 40(20) = 800$ and $\text{Var}[Y_{20}] = 100(20) = 2000$, we can write

$$\begin{aligned} P[Y_{20} > 1000] &= P\left[\frac{Y_{20}-800}{\sqrt{2000}} > \frac{1000-800}{\sqrt{2000}}\right] \\ &= P\left[Z > \frac{200}{20\sqrt{5}}\right] = Q(4.47) = 3.91 \times 10^{-6}. \end{aligned} \tag{1}$$

The second part is a little trickier. Since $E[Y_{25}] = 1000$, we know that the prof will spend around \$1000 in roughly 25 years. However, to be certain with probability 0.99 that the prof spends \$1000 will require more than 25 years. In particular, we know that

$$\begin{aligned} P[Y_n > 1000] &= P\left[\frac{Y_n - 40n}{\sqrt{100n}} > \frac{1000-40n}{\sqrt{100n}}\right] \\ &= 1 - \Phi\left(\frac{100-4n}{\sqrt{n}}\right) = 0.99. \end{aligned} \tag{2}$$

Hence, we must find n such that

$$\Phi\left(\frac{100-4n}{\sqrt{n}}\right) = 0.01. \tag{3}$$

Recall that $\Phi(x) = 0.01$ for a negative value of x. This is consistent with our earlier observation that we would need $n > 25$ corresponding to $100 - 4n < 0$. Thus, we use the identity $\Phi(x) = 1 - \Phi(-x)$ to write

$$\Phi\left(\frac{100 - 4n}{\sqrt{n}}\right) = 1 - \Phi\left(\frac{4n - 100}{\sqrt{n}}\right) = 0.01 \tag{4}$$

Equivalently, we have

$$\Phi\left(\frac{4n - 100}{\sqrt{n}}\right) = 0.99 \tag{5}$$

From the table of the Φ function, we have that $(4n - 100)/\sqrt{n} = 2.33$, or

$$(n - 25)^2 = (0.58)^2 n = 0.3393n. \tag{6}$$

Solving this quadratic yields $n = 28.09$. Hence, only after 28 years are we 99 percent sure that the prof will have spent \$1000. Note that a second root of the quadratic yields $n = 22.25$. This root is not a valid solution to our problem. Mathematically, it is a solution of our quadratic in which we choose the negative root of $\sqrt{n}$. This would correspond to assuming the standard deviation of Y_n is negative.