Probability and Stochastic Processes:
A Friendly Introduction for Electrical and Computer Engineers
Edition 3

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Yates and Goodman 3e Solution Set: 4.2.2, 4.2.4, 4.3.2, 4.3.4, 4.3.6, 4.4.2, 4.4.4, 4.4.6, and 4.4.7

Problem 4.2.2 Solution

The CDF of V was given to be

$$F_V(v) = \begin{cases} 0 & v < -5, \\ c(v+5)^2 & -5 \le v < 7, \\ 1 & v \ge 7. \end{cases} \tag{1}$$

(a) For V to be a continuous random variable, $F_V(v)$ must be a continuous function. This occurs if we choose c such that $F_V(v)$ doesn't have a discontinuity at $v = 7$. We meet this requirement if $c(7+5)^2 = 1$. This implies $c = 1/144$.

(b)

$$P[V > 4] = 1 - P[V \le 4] = 1 - F_V(4) = 1 - 81/144 = 63/144. \tag{2}$$

(c)

$$P[-3 < V \le 0] = F_V(0) - F_V(-3) = 25/144 - 4/144 = 21/144. \tag{3}$$

(d) Since $0 \le F_V(v) \le 1$ and since $F_V(v)$ is a nondecreasing function, it must be that $-5 \le a \le 7$. In this range,

$$P[V > a] = 1 - F_V(a) = 1 - (a+5)^2/144 = 2/3. \tag{4}$$

The unique solution in the range $-5 \le a \le 7$ is $a = 4\sqrt{3} - 5 = 1.928$.

Problem 4.2.4 Solution

In this problem, the CDF of W is

$$F_W(w) = \begin{cases} 0 & w < -5, \\ (w+5)/8 & -5 \leq w < -3, \\ 1/4 & -3 \leq w < 3, \\ 1/4 + 3(w-3)/8 & 3 \leq w < 5, \\ 1 & w \geq 5. \end{cases} \tag{1}$$

Each question can be answered directly from this CDF.

(a)

$$P[W \leq 4] = F_W(4) = 1/4 + 3/8 = 5/8. \tag{2}$$

(b)

$$P[-2 < W \leq 2] = F_W(2) - F_W(-2) = 1/4 - 1/4 = 0. \tag{3}$$

(c)

$$P[W > 0] = 1 - P[W \leq 0] = 1 - F_W(0) = 3/4. \tag{4}$$

(d) By inspection of $F_W(w)$, we observe that $P[W \leq a] = F_W(a) = 1/2$ for a in the range $3 \leq a \leq 5$. In this range,

$$F_W(a) = 1/4 + 3(a-3)/8 = 1/2. \tag{5}$$

This implies $a = 11/3$.

Problem 4.3.2 Solution

From the CDF, we can find the PDF by direct differentiation. The CDF and corresponding PDF are

$$F_X(x) = \begin{cases} 0 & x < -1, \\ (x+1)/2 & -1 \leq x \leq 1, \\ 1 & x > 1, \end{cases} \tag{1}$$

$$f_X(x) = \begin{cases} 1/2 & -1 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Problem 4.3.4 Solution

For $x < 0$, $F_X(x) = 0$. For $x \geq 0$,

$$F_X(x) = \int_0^x f_X(y)\, dy$$
$$= \int_0^x a^2 y e^{-a^2y^2/2}\, dy = -e^{-a^2y^2/2}\Big|_0^x = 1 - e^{-a^2x^2/2}. \tag{1}$$

A complete expression for the CDF of X is

$$F_X(x) = \begin{cases} 0 & x < 0, \\ 1 - e^{-a^2x^2/2} & x \geq 0 \end{cases} \tag{2}$$

Problem 4.3.6 Solution

$$f_X(x) = \begin{cases} ax^2 + bx & 0 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

First, we note that a and b must be chosen such that the above PDF integrates to 1.

$$\int_0^1 (ax^2 + bx)\, dx = a/3 + b/2 = 1 \tag{2}$$

Hence, $b = 2 - 2a/3$ and our PDF becomes

$$f_X(x) = x(ax + 2 - 2a/3) \tag{3}$$

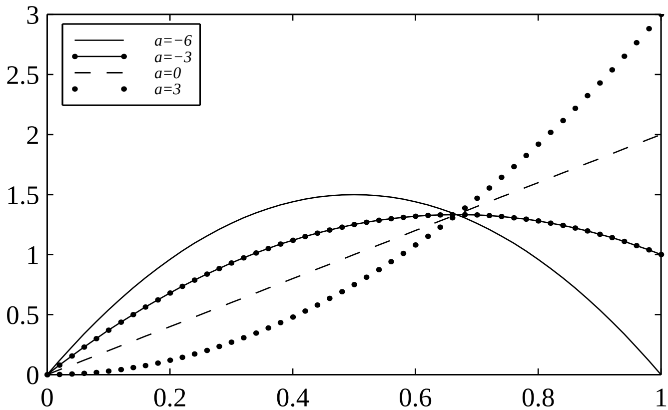
For the PDF to be non-negative for $x \in [0, 1]$, we must have $ax + 2 - 2a/3 \geq 0$ for all $x \in [0, 1]$. This requirement can be written as

$$a(2/3 - x) \leq 2, \qquad 0 \leq x \leq 1. \tag{4}$$

For $x = 2/3$, the requirement holds for all a. However, the problem is tricky because we must consider the cases $0 \leq x < 2/3$ and $2/3 < x \leq 1$ separately because of the sign change of the inequality. When $0 \leq x < 2/3$, we have $2/3 - x > 0$ and the requirement is most stringent at $x = 0$ where we require $2a/3 \leq 2$ or $a \leq 3$. When $2/3 < x \leq 1$, we can write the constraint as $a(x - 2/3) \geq -2$. In this case, the constraint is most stringent at $x = 1$, where we must have $a/3 \geq -2$ or $a \geq -6$. Thus a complete expression for our requirements are

$$-6 \leq a \leq 3, \qquad b = 2 - 2a/3. \tag{5}$$

As we see in the following plot, the shape of the PDF $f_X(x)$ varies greatly with the value of a.



Problem 4.4.2 Solution

(a) Since the PDF is uniform over [1,9]

$$\mathrm{E}[X] = \frac{1+9}{2} = 5, \qquad \mathrm{Var}[X] = \frac{(9-1)^2}{12} = \frac{16}{3}. \tag{1}$$

(b) Define $h(X) = 1/\sqrt{X}$ then

$$h(\mathrm{E}[X]) = 1/\sqrt{5}, \tag{2}$$

$$\mathrm{E}[h(X)] = \int_1^9 \frac{x^{-1/2}}{8}\,dx = 1/2. \tag{3}$$

(c)

$$\mathrm{E}[Y] = \mathrm{E}[h(X)] = 1/2, \tag{4}$$

$$\begin{aligned} \mathrm{Var}[Y] &= \mathrm{E}\left[Y^2\right] - (\mathrm{E}[Y])^2 \\ &= \int_1^9 \frac{x^{-1}}{8}\,dx - \mathrm{E}[X]^2 = \frac{\ln 9}{8} - 1/4. \end{aligned} \tag{5}$$

Problem 4.4.4 Solution

We can find the expected value of X by direct integration of the given PDF.

$$f_Y(y) = \begin{cases} y/2 & 0 \le y \le 2, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The expectation is

$$\mathrm{E}[Y] = \int_0^2 \frac{y^2}{2}\,dy = 4/3. \tag{2}$$

To find the variance, we first find the second moment

$$\mathrm{E}\left[Y^2\right] = \int_0^2 \frac{y^3}{2}\,dy = 2. \tag{3}$$

The variance is then $\mathrm{Var}[Y] = \mathrm{E}[Y^2] - \mathrm{E}[Y]^2 = 2 - (4/3)^2 = 2/9$.

Problem 4.4.6 Solution

To evaluate the moments of V, we need the PDF $f_V(v)$, which we find by taking the derivative of the CDF $F_V(v)$. The CDF and corresponding PDF of V are

$$F_V(v) = \begin{cases} 0 & v < -5, \\ (v+5)^2/144 & -5 \le v < 7, \\ 1 & v \ge 7, \end{cases} \tag{1}$$

$$f_V(v) = \begin{cases} 0 & v < -5, \\ (v+5)/72 & -5 \le v < 7, \\ 0 & v \ge 7. \end{cases} \tag{2}$$

(a) The expected value of V is

$$\begin{aligned}
\mathrm{E}\left[V\right] = \int_{-\infty}^{\infty} v f_V(v)\ dv &= \frac{1}{72}\int_{-5}^{7}(v^2+5v)\,dv \\
&= \frac{1}{72}\left(\frac{v^3}{3}+\frac{5v^2}{2}\right)\Bigg|_{-5}^{7} \\
&= \frac{1}{72}\left(\frac{343}{3}+\frac{245}{2}+\frac{125}{3}-\frac{125}{2}\right) = 3.
\end{aligned} \tag{3}$$

(b) To find the variance, we first find the second moment

$$\begin{aligned}
\mathrm{E}\left[V^2\right] = \int_{-\infty}^{\infty} v^2 f_V(v)\ dv &= \frac{1}{72}\int_{-5}^{7}(v^3+5v^2)\,dv \\
&= \frac{1}{72}\left(\frac{v^4}{4}+\frac{5v^3}{3}\right)\Bigg|_{-5}^{7} \\
&= 6719/432 = 15.55.
\end{aligned} \tag{4}$$

The variance is $\mathrm{Var}[V] = \mathrm{E}[V^2] - (\mathrm{E}[V])^2 = 2831/432 = 6.55$.

(c) The third moment of V is

$$\begin{aligned}
\mathrm{E}\left[V^3\right] &= \int_{-\infty}^{\infty} v^3 f_V(v)\ dv \\
&= \frac{1}{72}\int_{-5}^{7}(v^4+5v^3)\,dv \\
&= \frac{1}{72}\left(\frac{v^5}{5}+\frac{5v^4}{4}\right)\Bigg|_{-5}^{7} = 86.2.
\end{aligned} \tag{5}$$

Problem 4.4.7 Solution

To find the moments, we first find the PDF of U by taking the derivative of $F_U(u)$.

The CDF and corresponding PDF are

$$F_U(u) = \begin{cases} 0 & u < -5, \\ (u+5)/8 & -5 \le u < -3, \\ 1/4 & -3 \le u < 3, \\ 1/4 + 3(u-3)/8 & 3 \le u < 5, \\ 1 & u \ge 5. \end{cases} \tag{1}$$

$$f_U(u) = \begin{cases} 0 & u < -5, \\ 1/8 & -5 \le u < -3, \\ 0 & -3 \le u < 3, \\ 3/8 & 3 \le u < 5, \\ 0 & u \ge 5. \end{cases} \tag{2}$$

(a) The expected value of U is

$$\begin{aligned} \mathrm{E}[U] = \int_{-\infty}^{\infty} u f_U(u)\, du &= \int_{-5}^{-3} \frac{u}{8}\, du + \int_{3}^{5} \frac{3u}{8}\, du \\ &= \left.\frac{u^2}{16}\right|_{-5}^{-3} + \left.\frac{3u^2}{16}\right|_{3}^{5} = 2. \end{aligned} \tag{3}$$

(b) The second moment of U is

$$\begin{aligned} \mathrm{E}\left[U^2\right] = \int_{-\infty}^{\infty} u^2 f_U(u)\, du &= \int_{-5}^{-3} \frac{u^2}{8}\, du + \int_{3}^{5} \frac{3u^2}{8}\, du \\ &= \left.\frac{u^3}{24}\right|_{-5}^{-3} + \left.\frac{u^3}{8}\right|_{3}^{5} = 49/3. \end{aligned} \tag{4}$$

The variance of U is $\mathrm{Var}[U] = \mathrm{E}[U^2] - (\mathrm{E}[U])^2 = 37/3$.

(c) Note that $2^U = e^{(\ln 2)U}$. This implies that

$$\int 2^u\, du = \int e^{(\ln 2)u}\, du = \frac{1}{\ln 2} e^{(\ln 2)u} = \frac{2^u}{\ln 2}. \tag{5}$$

The expected value of 2^U is then

$$
\begin{aligned}
\mathrm{E}\left[2^U\right] &= \int_{-\infty}^{\infty} 2^u f_U(u)\ du \\
&= \int_{-5}^{-3} \frac{2^u}{8}\, du + \int_{3}^{5} \frac{3 \cdot 2^u}{8}\, du \\
&= \left.\frac{2^u}{8 \ln 2}\right|_{-5}^{-3} + \left.\frac{3 \cdot 2^u}{8 \ln 2}\right|_{3}^{5} = \frac{2307}{256 \ln 2} = 13.001.
\end{aligned} \tag{6}
$$