Probability and Stochastic Processes:
A Friendly Introduction for Electrical and Computer Engineers
Edition 3
Roy D. Yates and David J. Goodman

Yates and Goodman 3e Solution Set: 3.5.2, 3.5.3, 3.5.5, 3.5.7, 3.5.12, 3.5.14, 3.8.1, 3.8.4, 3.8.6, and 3.8.8

Problem 3.5.2 Solution

Voice calls and data calls each cost 20 cents and 30 cents respectively. Furthermore the respective probabilities of each type of call are 0.6 and 0.4.

(a) Since each call is either a voice or data call, the cost of one call can only take the two values associated with the cost of each type of call. Therefore the PMF of X is

$$P_X(x) = \begin{cases} 0.6 & x = 20, \\ 0.4 & x = 30, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(b) The expected cost, $\mathrm{E}[C]$, is simply the sum of the cost of each type of call multiplied by the probability of such a call occurring.

$$\mathrm{E}[C] = 20(0.6) + 30(0.4) = 24 \text{ cents.} \tag{2}$$

Problem 3.5.3 Solution

(a) J has the Poisson PMF

$$P_J(j) = \begin{cases} t^j e^{-t}/j! & j = 0, 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

It follows that

$$0.9 = \mathrm{P}[J > 0] = 1 - P_J(0) = 1 - e^{-t} \qquad \Longrightarrow t = \ln(10) = 2.3. \tag{2}$$

(b) For $k = 0, 1, 2, \ldots$, $P_K(k) = 10^k e^{-10}/k!$. Thus

$$P[K = 10] = P_K(10) = 10^{10} e^{-10} = 0.1251. \tag{3}$$

(c) L is a Poisson ($\alpha = E[L] = 2$) random variable. Thus its PMF is

$$P_L(l) = \begin{cases} 2^l e^{-2}/l! & l = 0, 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

It follows that

$$P[L \le 1] = P_L(0) + P_L(1) = 3e^{-2} = 0.406. \tag{5}$$

Problem 3.5.5 Solution

(a) Each packet transmission is a Bernoulli trial with success probability 0.95 and X is the number of packet failures (received in error) in 10 trials. Since the failure probability is $p = 0.05$, X has the binomial ($n = 10, p = 0.05$) PMF

$$P_X(x) = \binom{10}{x} (0.05)^x (0.95)^{10-x}. \tag{1}$$

(b) When you send 12 thousand packets, the number of packets received in error, Y, is a binomial ($n = 12000, p = 0.05$) random variable. The expected number received in error is $E[Y] = np = 600$ per hour, or about 10 packets per minute. Keep in mind this is a reasonable figure if you are an active data user.

Problem 3.5.7 Solution

From the solution to Problem 3.4.2, the PMF of X is

$$P_X(x) = \begin{cases} 0.2 & x = -1, \\ 0.5 & x = 0, \\ 0.3 & x = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The expected value of X is

$$E[X] = \sum_x x P_X(x) = -1(0.2) + 0(0.5) + 1(0.3) = 0.1. \tag{2}$$

Problem 3.5.12 Solution

(a) The probability of 20 heads in a row is $p = (1/2)^{20} = 2^{-20}$. Hence the PMF of R is

$$P_R(r) = \begin{cases} 1 - 2^{-20} & r = 0, \\ 2^{-20} & r = 20 \cdot 10^6, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(b) Let's say a success occurs if a contestant wins the game. The success probability is $p = 2^{-20}$. Note that there are $L = l$ losing customers if we observe l failures followed by a success on trial $l + 1$. Note that $L = 0$ is possible. Thus $P[L = l] = (1 - p)^l p$. The PMF of L is

$$P_L(l) = \begin{cases} (1 - p)^l p & l = 0, 1, 2, \ldots \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

(c) The expected reward earned by a customer is

$$E[R] = 20 \cdot 10^6 p = \frac{20 \cdot 10^6}{2^{20}}. \tag{3}$$

You might know that $2^{10} = 1024$ and so $2^{20} = (1024)^2 > 10^6$. Thus $E[R] < 20$. In fact, $E[R] = 19.07$. That is, the casino collects \$20 from each player but on average pays out \$19.07 to each customer.

Problem 3.5.14 Solution

(a) When K is geometric ($p = 1/3$), it has PMF

$$P_K(k) = \begin{cases} (1-p)^{k-1}p & k = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

and $\mathrm{E}[K] = 1/p = 3$. In this case,

$$\begin{aligned} \mathrm{P}\left[K < \mathrm{E}\left[K\right]\right] = \mathrm{P}\left[K < 3\right] &= P_K(1) + P_K(2) \\ &= p + (1-p)p = 5/9 = 0.555. \end{aligned} \tag{2}$$

(b) When K is binomial ($n = 6, p = 1/2$), it has PMF

$$P_K(k) = \binom{6}{k} p^k (1-p)^{6-k} = \binom{6}{k} \left(\frac{1}{2}\right)^k \tag{3}$$

and $\mathrm{E}[K] = np = 3$. In this case,

$$\begin{aligned} \mathrm{P}\left[K < \mathrm{E}\left[K\right]\right] = \mathrm{P}\left[K < 3\right] &= P_K(0) + P_K(1) + P_K(2) \\ &= \left(\frac{1}{2}\right)^6 (1 + 6 + 15) \\ &= \frac{22}{64} = \frac{11}{32} = 0.344. \end{aligned} \tag{4}$$

(c) When K is Poisson ($\alpha = 3$), it has PMF

$$\begin{aligned} P_K(k) &= \begin{cases} \alpha^k e^{-\alpha}/k! & k = 0, 1, 2, \ldots, \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} 3^k e^{-3}/k! & k = 0, 1, 2, \ldots, \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \tag{5}$$

and $\mathrm{E}[K] = \alpha = 3$. In this case,

$$\begin{aligned} \mathrm{P}\left[K < \mathrm{E}\left[K\right]\right] = \mathrm{P}\left[K < 3\right] &= P_K(0) + P_K(1) + P_K(2) \\ &= e^{-3}\left(1 + 3 + \frac{3^2}{3!}\right) \\ &= (11/2)e^{-3} = 0.2738. \end{aligned} \tag{6}$$

(d) When K is discrete uniform $(0,6)$, it has PMF

$$P_K(k) = \begin{cases} 1/7 & k = 0,1,\ldots,6, \\ 0 & \text{otherwise}, \end{cases} \tag{7}$$

and $\mathrm{E}[K] = (0+6)/2 = 3$. In this case,

$$\begin{aligned} \mathrm{P}\left[K < \mathrm{E}\left[K\right]\right] = \mathrm{P}\left[K < 3\right] &= P_K(0) + P_K(1) + P_K(2) \\ &= 3/7 = 0.429. \end{aligned} \tag{8}$$

Problem 3.8.1 Solution

Given the following PMF

$$P_N(n) = \begin{cases} 0.2 & n = 0, \\ 0.7 & n = 1, \\ 0.1 & n = 2, \\ 0 & \text{otherwise}, \end{cases} \tag{1}$$

the calculations are straightforward:

(a) $\mathrm{E}[N] = (0.2)0 + (0.7)1 + (0.1)2 = 0.9$.

(b) $\mathrm{E}[N^2] = (0.2)0^2 + (0.7)1^2 + (0.1)2^2 = 1.1$.

(c) $\mathrm{Var}[N] = \mathrm{E}[N^2] - \mathrm{E}[N]^2 = 1.1 - (0.9)^2 = 0.29$.

(d) $\sigma_N = \sqrt{\mathrm{Var}[N]} = \sqrt{0.29}$.

Problem 3.8.4 Solution

From the solution to Problem 3.4.3, the PMF of X is

$$P_X(x) = \begin{cases} 0.4 & x = -3, \\ 0.4 & x = 5, \\ 0.2 & x = 7, \\ 0 & \text{otherwise}. \end{cases} \tag{1}$$

The expected value of X is

$$\mathrm{E}[X] = \sum_x x P_X(x) = -3(0.4) + 5(0.4) + 7(0.2) = 2.2. \tag{2}$$

The expected value of X^2 is

$$\mathrm{E}\left[X^2\right] = \sum_x x^2 P_X(x) = (-3)^2(0.4) + 5^2(0.4) + 7^2(0.2) = 23.4. \tag{3}$$

The variance of X is

$$\mathrm{Var}[X] = \mathrm{E}\left[X^2\right] - (\mathrm{E}[X])^2 = 23.4 - (2.2)^2 = 18.56. \tag{4}$$

Problem 3.8.6 Solution

(a) The expected value of X is

$$\begin{aligned}
\mathrm{E}[X] &= \sum_{x=0}^{5} x P_X(x) \\
&= 0\binom{5}{0}\frac{1}{2^5} + 1\binom{5}{1}\frac{1}{2^5} + 2\binom{5}{2}\frac{1}{2^5} \\
&\qquad + 3\binom{5}{3}\frac{1}{2^5} + 4\binom{5}{4}\frac{1}{2^5} + 5\binom{5}{5}\frac{1}{2^5} \\
&= [5 + 20 + 30 + 20 + 5]/2^5 = 5/2.
\end{aligned} \tag{1}$$

The expected value of X^2 is

$$\begin{aligned}
\mathrm{E}\left[X^2\right] &= \sum_{x=0}^{5} x^2 P_X(x) \\
&= 0^2\binom{5}{0}\frac{1}{2^5} + 1^2\binom{5}{1}\frac{1}{2^5} + 2^2\binom{5}{2}\frac{1}{2^5} \\
&\qquad + 3^2\binom{5}{3}\frac{1}{2^5} + 4^2\binom{5}{4}\frac{1}{2^5} + 5^2\binom{5}{5}\frac{1}{2^5} \\
&= [5 + 40 + 90 + 80 + 25]/2^5 = 240/32 = 15/2.
\end{aligned} \tag{2}$$

The variance of X is

$$\text{Var}[X] = \text{E}\left[X^2\right] - (\text{E}\left[X\right])^2 = 15/2 - 25/4 = 5/4. \tag{3}$$

By taking the square root of the variance, the standard deviation of X is $\sigma_X = \sqrt{5/4} \approx 1.12$.

(b) The probability that X is within one standard deviation of its mean is

$$\begin{aligned} \text{P}\left[\mu_X - \sigma_X \leq X \leq \mu_X + \sigma_X\right] &= \text{P}\left[2.5 - 1.12 \leq X \leq 2.5 + 1.12\right] \\ &= \text{P}\left[1.38 \leq X \leq 3.62\right] \\ &= \text{P}\left[2 \leq X \leq 3\right]. \end{aligned} \tag{4}$$

By summing the PMF over the desired range, we obtain

$$\begin{aligned} \text{P}\left[2 \leq X \leq 3\right] &= P_X(2) + P_X(3) \\ &= 10/32 + 10/32 = 5/8. \end{aligned} \tag{5}$$

Problem 3.8.8 Solution

Given that

$$Y = \frac{1}{\sigma_x}(X - \mu_X), \tag{1}$$

we can use the linearity property of the expectation operator to find the mean value

$$\text{E}\left[Y\right] = \frac{\text{E}\left[X - \mu_X\right]}{\sigma_X} = \frac{\text{E}\left[X\right] - \text{E}\left[X\right]}{\sigma_X} = 0. \tag{2}$$

Using the fact that $\text{Var}[aX + b] = a^2 \text{Var}[X]$, the variance of Y is found to be

$$\text{Var}\left[Y\right] = \frac{1}{\sigma_X^2} \text{Var}\left[X\right] = 1. \tag{3}$$