Probability and Stochastic Processes:
A Friendly Introduction for Electrical and Computer Engineers
Edition 3
Roy D. Yates and David J. Goodman

Yates and Goodman 3e Solution Set: 12.1.3, 12.1.4, 12.1.5, 12.2.1, 12.2.4, and 12.2.5

Problem 12.1.3 Solution

(a) For $0 \leq x \leq 1$,

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy = \int_x^1 2\, dy = 2(1 - x). \tag{1}$$

The complete expression of the PDF of X is

$$f_X(x) = \begin{cases} 2(1 - x) & 0 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

(b) The blind estimate of X is

$$\hat{X}_B = \mathrm{E}[X] = \int_0^1 2x(1 - x)\, dx = \left(x^2 - \frac{2x^3}{3}\right)\Bigg|_0^1 = \frac{1}{3}. \tag{3}$$

(c) First we calculate

$$\begin{aligned} \mathrm{P}[X > 1/2] &= \int_{1/2}^1 f_X(x)\, dx \\ &= \int_{1/2}^1 2(1 - x)\, dx = (2x - x^2)\big|_{1/2}^1 = \frac{1}{4}. \end{aligned} \tag{4}$$

Now we calculate the conditional PDF of X given $X > 1/2$.

$$f_{X|X>1/2}(x) = \begin{cases} \frac{f_X(x)}{\mathrm{P}[X>1/2]} & x > 1/2, \\ 0 & \text{otherwise,} \end{cases}$$
$$= \begin{cases} 8(1-x) & 1/2 < x \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

The minimum mean square error estimate of X given $X > 1/2$ is

$$\mathrm{E}[X|X > 1/2] = \int_{-\infty}^{\infty} x f_{X|X>1/2}(x)\, dx$$
$$= \int_{1/2}^{1} 8x(1-x)\, dx = \left(4x^2 - \frac{8x^3}{3}\right)\Big|_{1/2}^{1} = \frac{2}{3}. \tag{6}$$

(d) For $0 \leq y \leq 1$, the marginal PDF of Y is

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dx = \int_0^y 2\, dx = 2y. \tag{7}$$

The complete expression for the marginal PDF of Y is

$$f_Y(y) = \begin{cases} 2y & 0 \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

(e) The blind estimate of Y is

$$\hat{y}_B = \mathrm{E}[Y] = \int_0^1 2y^2\, dy = \frac{2}{3}. \tag{9}$$

(f) We already know that $\mathrm{P}[X > 1/2] = 1/4$. However, this problem differs from the other problems in this section because we will estimate Y based on the observation of X. In this case, we need to calculate the conditional joint PDF

$$f_{X,Y|X>1/2}(x, y) = \begin{cases} \frac{f_{X,Y}(x,y)}{\mathrm{P}[X>1/2]} & x > 1/2, \\ 0 & \text{otherwise,} \end{cases}$$
$$= \begin{cases} 8 & 1/2 < x \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

The MMSE estimate of Y given $X > 1/2$ is

$$\begin{aligned} \mathrm{E}\left[Y|X > 1/2\right] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f_{X,Y|X>1/2}(x,y)\ dx\, dy \\ &= \int_{1/2}^{1} y\left(\int_{1/2}^{y} 8\, dx\right)\ dy \\ &= \int_{1/2}^{1} y(8y-4)\, dy = \frac{5}{6}. \end{aligned} \tag{11}$$

Problem 12.1.4 Solution

The joint PDF of X and Y is

$$f_{X,Y}(x,y) = \begin{cases} 6(y-x) & 0 \le x \le y \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(a) The conditional PDF of X given Y is found by dividing the joint PDF by the marginal with respect to Y. For $y < 0$ or $y > 1$, $f_Y(y) = 0$. For $0 \le y \le 1$,

$$f_Y(y) = \int_0^y 6(y-x)\, dx = 6xy - 3x^2\Big|_0^y = 3y^2 \tag{2}$$

The complete expression for the marginal PDF of Y is

$$f_Y(y) = \begin{cases} 3y^2 & 0 \le y \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Thus for $0 < y \le 1$,

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \frac{6(y-x)}{3y^2} & 0 \le x \le y, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

(b) The minimum mean square estimator of X given $Y = y$ is

$$\hat{X}_M(y) = \mathrm{E}\left[X|Y = y\right] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\, dx$$
$$= \int_0^y \frac{6x(y-x)}{3y^2}\, dx$$
$$= \left. \frac{3x^2y - 2x^3}{3y^2} \right|_{x=0}^{x=y} = y/3. \tag{5}$$

(c) First we must find the marginal PDF for X. For $0 \le x \le 1$,

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy = \int_x^1 6(y-x)\, dy = 3y^2 - 6xy\Big|_{y=x}^{y=1}$$
$$= 3(1 - 2x + x^2) = 3(1-x)^2. \tag{6}$$

The conditional PDF of Y given X is

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \begin{cases} \frac{2(y-x)}{1-2x+x^2} & x \le y \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

(d) The minimum mean square estimator of Y given X is

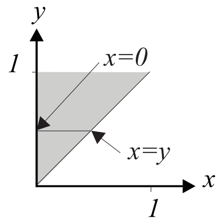
$$\hat{Y}_M(x) = \mathrm{E}\left[Y|X = x\right] = \int_{-\infty}^{\infty} y f_{Y|X}(y|x)\, dy$$
$$= \int_x^1 \frac{2y(y-x)}{1-2x+x^2}\, dy$$
$$= \left. \frac{(2/3)y^3 - y^2x}{1-2x+x^2} \right|_{y=x}^{y=1} = \frac{2 - 3x + x^3}{3(1-x)^2}. \tag{8}$$

Perhaps surprisingly, this result can be simplified to

$$\hat{Y}_M(x) = \frac{x}{3} + \frac{2}{3}. \tag{9}$$

Problem 12.1.5 Solution

(a) First we find the marginal PDF $f_Y(y)$. For $0 \le y \le 2$,



$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx = \int_0^y 2\, dx = 2y. \tag{1}$$

Hence, for $0 \le y \le 2$, the conditional PDF of X given Y is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} 1/y & 0 \le x \le y, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

(b) The optimum mean squared error estimate of X given $Y = y$ is

$$\hat{x}_M(y) = \mathrm{E}\left[X|Y=y\right] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\, dx = \int_0^y \frac{x}{y}\, dx = y/2. \tag{3}$$

(c) The MMSE estimator of X given Y is $\hat{X}_M(Y) = \mathrm{E}[X|Y] = Y/2$. The mean squared error is

$$\begin{aligned} e^*_{X,Y} &= \mathrm{E}\left[(X - \hat{X}_M(Y))^2\right] \\ &= \mathrm{E}\left[(X - Y/2)^2\right] = \mathrm{E}\left[X^2 - XY + Y^2/4\right]. \end{aligned} \tag{4}$$

Of course, the integral must be evaluated.

$$\begin{aligned} e^*_{X,Y} &= \int_0^1 \int_0^y 2(x^2 - xy + y^2/4)\, dx\, dy \\ &= \int_0^1 \left(2x^3/3 - x^2 y + xy^2/2\right)\Big|_{x=0}^{x=y} dy \\ &= \int_0^1 \frac{y^3}{6}\, dy = 1/24. \end{aligned} \tag{5}$$

Another approach to finding the mean square error is to recognize that the MMSE estimator is a linear estimator and thus must be the optimal linear estimator. Hence, the mean square error of the optimal linear estimator given by Theorem 12.3 must equal $e^*_{X,Y}$. That is, $e^*_{X,Y} = \text{Var}[X](1 - \rho^2_{X,Y})$. However, calculation of the correlation coefficient $\rho_{X,Y}$ is at least as much work as direct calculation of $e^*_{X,Y}$.

Problem 12.2.1 Solution

(a) The marginal PMFs of X and Y are listed below

$$P_X(x) = \begin{cases} 1/3 & x = -1, 0, 1, \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

$$P_Y(y) = \begin{cases} 1/4 & y = -3, -1, 0, 1, 3, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

(b) No, the random variables X and Y are not independent since

$$P_{X,Y}(1, -3) = 0 \neq P_X(1)\, P_Y(-3) \tag{3}$$

(c) Direct evaluation leads to

$$\mathrm{E}[X] = 0, \qquad \text{Var}[X] = 2/3, \tag{4}$$

$$\mathrm{E}[Y] = 0, \qquad \text{Var}[Y] = 5. \tag{5}$$

This implies

$$\text{Cov}[X, Y] = \mathrm{E}[XY] - \mathrm{E}[X]\,\mathrm{E}[Y] = \mathrm{E}[XY] = 7/6. \tag{6}$$

(d) From Theorem 12.3, the optimal linear estimate of X given Y is

$$\hat{X}_L(Y) = \rho_{X,Y}\frac{\sigma_X}{\sigma_Y}(Y - \mu_Y) + \mu_X = \frac{7}{30}Y + 0. \tag{7}$$

Therefore, $a^* = 7/30$ and $b^* = 0$.

(e) From the previous part, X and Y have correlation coefficient

$$\rho_{X,Y} = \mathrm{Cov}\left[X,Y\right]/\sqrt{\mathrm{Var}[X]\,\mathrm{Var}[Y]} = \sqrt{49/120}. \tag{8}$$

From Theorem 12.3, the minimum mean square error of the optimum linear estimate is

$$e_L^* = \sigma_X^2(1-\rho_{X,Y}^2) = \frac{2}{3}\frac{71}{120} = \frac{71}{180}. \tag{9}$$

(f) The conditional probability mass function is

$$P_{X|Y}(x|-3) = \frac{P_{X,Y}(x,-3)}{P_Y(-3)} = \begin{cases} \frac{1/6}{1/4} = 2/3 & x=-1, \\ \frac{1/12}{1/4} = 1/3 & x=0, \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

(g) The minimum mean square estimator of X given that $Y=3$ is

$$\hat{x}_M(-3) = \mathrm{E}\left[X|Y=-3\right] = \sum_x x P_{X|Y}(x|-3) = -2/3. \tag{11}$$

(h) The mean squared error of this estimator is

$$\begin{aligned} \hat{e}_M(-3) &= \mathrm{E}\left[(X-\hat{x}_M(-3))^2|Y=-3\right] \\ &= \sum_x (x+2/3)^2 P_{X|Y}(x|-3) \\ &= (-1/3)^2(2/3) + (2/3)^2(1/3) = 2/9. \end{aligned} \tag{12}$$

Problem 12.2.4 Solution

To solve this problem, we use Theorem 12.3. The only difficulty is in computing $\mathrm{E}[X]$, $\mathrm{E}[Y]$, $\mathrm{Var}[X]$, $\mathrm{Var}[Y]$, and $\rho_{X,Y}$. First we calculate the marginal PDFs

$$f_X(x) = \int_x^1 2(y+x)\,dy = y^2 + 2xy\Big|_{y=x}^{y=1} = 1 + 2x - 3x^2, \tag{1}$$

$$f_Y(y) = \int_0^y 2(y+x)\,dx = 2xy + x^2\Big|_{x=0}^{x=y} = 3y^2. \tag{2}$$

The first and second moments of X are

$$\mathrm{E}[X] = \int_0^1 (x + 2x^2 - 3x^3)\,dx = x^2/2 + 2x^3/3 - 3x^4/4\Big|_0^1 = 5/12, \tag{3}$$

$$\mathrm{E}\left[X^2\right] = \int_0^1 (x^2 + 2x^3 - 3x^4)\,dx = x^3/3 + x^4/2 - 3x^5/5\Big|_0^1 = 7/30. \tag{4}$$

The first and second moments of Y are

$$\mathrm{E}[Y] = \int_0^1 3y^3\,dy = 3/4, \qquad \mathrm{E}\left[Y^2\right] = \int_0^1 3y^4\,dy = 3/5. \tag{5}$$

Thus, X and Y each have variance

$$\mathrm{Var}[X] = \mathrm{E}\left[X^2\right] - (\mathrm{E}[X])^2 = \frac{129}{2160}, \tag{6}$$

$$\mathrm{Var}[Y] = \mathrm{E}\left[Y^2\right] - (\mathrm{E}[Y])^2 = \frac{3}{80}. \tag{7}$$

To calculate the correlation coefficient, we first must calculate the correlation

$$\begin{aligned}\mathrm{E}[XY] &= \int_0^1 \int_0^y 2xy(x+y)\,dx\,dy \\ &= \int_0^1 \left[2x^3y/3 + x^2y^2\right]\Big|_{x=0}^{x=y} dy = \int_0^1 \frac{5y^4}{3}\,dy = 1/3.\end{aligned} \tag{8}$$

Hence, the correlation coefficient is

$$\rho_{X,Y} = \frac{\mathrm{Cov}[X,Y]}{\sqrt{\mathrm{Var}[X]\,\mathrm{Var}[Y]}}\frac{\mathrm{E}[XY] - \mathrm{E}[X]\,\mathrm{E}[Y]}{\sqrt{\mathrm{Var}[X]\,\mathrm{Var}[Y]}} = \frac{5}{\sqrt{129}}. \tag{9}$$

Finally, we use Theorem 12.3 to combine these quantities in the optimal linear estimator.

$$\begin{aligned}\hat{X}_L(Y) &= \rho_{X,Y}\frac{\sigma_X}{\sigma_Y}(Y - \mathrm{E}[Y]) + \mathrm{E}[X] \\ &= \frac{5}{\sqrt{129}}\frac{\sqrt{129}}{9}\left(Y - \frac{3}{4}\right) + \frac{5}{12} = \frac{5}{9}Y.\end{aligned} \tag{10}$$

Problem 12.2.5 Solution

The linear mean square estimator of X given Y is

$$\hat{X}_L(Y) = \left(\frac{\mathrm{E}[XY] - \mu_X \mu_Y}{\mathrm{Var}[Y]}\right)(Y - \mu_Y) + \mu_X. \tag{1}$$

To find the parameters of this estimator, we calculate

$$f_Y(y) = \int_0^y 6(y-x)\,dx = 6xy - 3x^2\Big|_0^y = 3y^2 \qquad (0 \le y \le 1), \tag{2}$$

$$f_X(x) = \int_x^1 6(y-x)\,dy = \begin{cases} 3(1 + -2x + x^2) & 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

The moments of X and Y are

$$\mathrm{E}[Y] = \int_0^1 3y^3\,dy = 3/4, \tag{4}$$

$$\mathrm{E}[X] = \int_0^1 3x(1 - 2x + x^2)\,dx = 1/4, \tag{5}$$

$$\mathrm{E}\left[Y^2\right] = \int_0^1 3y^4\,dy = 3/5, \tag{6}$$

$$\mathrm{E}\left[X^2\right] = \int_0^1 3x^2(1 + -2x + x^2)\,dx = 1/10. \tag{7}$$

The correlation between X and Y is

$$\mathrm{E}[XY] = 6\int_0^1 \int_x^1 xy(y-x)\,dy\,dx = 1/5. \tag{8}$$

Putting these pieces together, the optimal linear estimate of X given Y is

$$\hat{X}_L(Y) = \left(\frac{1/5 - 3/16}{3/5 - (3/4)^2}\right)\left(Y - \frac{3}{4}\right) + \frac{1}{4} = \frac{Y}{3}. \tag{9}$$