Probability and Stochastic Processes:
A Friendly Introduction for Electrical and Computer Engineers
Edition 3
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Yates and Goodman 3e Solution Set: 11.2.1, 11.2.2, and 11.2.4

Problem 11.2.1 Solution

For the MAP test, we must choose acceptance regions A_0 and A_1 for the two hypotheses H_0 and H_1. From Theorem 11.2, the MAP rule is

$$n \in A_0 \text{ if } \frac{P_{N|H_0}(n)}{P_{N|H_1}(n)} \geq \frac{\mathrm{P}[H_1]}{\mathrm{P}[H_0]}; \qquad n \in A_1 \text{ otherwise.} \tag{1}$$

Since $P_{N|H_i}(n) = \lambda_i^n e^{-\lambda_i}/n!$, the MAP rule becomes

$$n \in A_0 \text{ if } \left(\frac{\lambda_0}{\lambda_1}\right)^n e^{-(\lambda_0-\lambda_1)} \geq \frac{\mathrm{P}[H_1]}{\mathrm{P}[H_0]}; \qquad n \in A_1 \text{ otherwise.} \tag{2}$$

By taking logarithms and assuming $\lambda_1 > \lambda_0$ yields the final form of the MAP rule

$$n \in A_0 \text{ if } n \leq n^* = \frac{\lambda_1 - \lambda_0 + \ln(\mathrm{P}[H_0]/\mathrm{P}[H_1])}{\ln(\lambda_1/\lambda_0)}; \qquad n \in A_1 \text{ otherwise.} \tag{3}$$

From the MAP rule, we can get the ML rule by setting the a priori probabilities to be equal. This yields the ML rule

$$n \in A_0 \text{ if } n \leq n^* = \frac{\lambda_1 - \lambda_0}{\ln(\lambda_1/\lambda_0)}; \qquad n \in A_1 \text{ otherwise.} \tag{4}$$

Problem 11.2.2 Solution

Hypotheses H_0 and H_1 have a priori probabilities $\mathrm{P}[H_0] = 0.8$ and $\mathrm{P}[H_1] = 0.2$ and likelihood functions

$$f_{T|H_0}(t) = \begin{cases} (1/3)e^{-t/3} & t \geq 0, \\ & \text{otherwise,} \end{cases}$$

$$f_{T|H_1}(t) = \begin{cases} (1/\mu_D)e^{-t/\mu_D} & t \geq 0, \\ & \text{otherwise.} \end{cases} \tag{1}$$

The acceptance regions are $A_0 = \{t | T \leq t_0\}$ and $A_1 = \{t | t > t_0\}$.

(a) The false alarm probability is

$$P_{\mathrm{FA}} = \mathrm{P}[A_1|H_0] = \int_{t_0}^{\infty} f_{T|H_0}(t)\ dt = e^{-t_0/3}. \tag{2}$$

(b) The miss probability is

$$P_{\mathrm{MISS}} = \mathrm{P}[A_0|H_1] = \int_{0}^{t_0} f_{T|H_1}(t)\ dt = 1 - e^{-t_0/\mu_D}. \tag{3}$$

(c) From Theorem 11.6, the maximum likelihood decision rule is

$$t \in A_0 \text{ if } \frac{f_{T|H_0}(t)}{f_{T|H_1}(t)} \geq 1; \qquad t \in A_1 \text{ otherwise.} \tag{4}$$

After some algebra, this rule simplifies to

$$t \in A_0 \text{ if } t \leq t_{ML} = \frac{\ln(\mu_D/3)}{1/3 - 1/\mu_D}; \qquad t \in A_1 \text{ otherwise.} \tag{5}$$

When $\mu_D = 6$ minutes, $t_{ML} = 6 \ln 2 = 4.16$ minutes. When $\mu_D = 10$ minutes, $t_{ML} = (30/7)\ln(10/3) = 5.16$ minutes.

(d) The ML rule is the same as the MAP rule when $\mathrm{P}[H_0] = \mathrm{P}[H_1]$. When $\mathrm{P}[H_0] > \mathrm{P}[H_1]$, the MAP rule (which minimizes the probability of an error) should enlarge the A_0 acceptance region. Thus we would expect $t_{\mathrm{MAP}} > t_{ML}$.

(e) From Theorem 11.2, the MAP rule is

$$t \in A_0 \text{ if } \frac{f_{T|H_0}(t)}{f_{T|H_1}(t)} \geq \frac{\mathrm{P}[H_1]}{\mathrm{P}[H_0]} = \frac{1}{4}; \qquad t \in A_1 \text{ otherwise.} \tag{6}$$

This rule simplifies to

$$t \in A_0 \text{ if } t \leq t_{\mathrm{MAP}} = \frac{\ln(4\mu_D/3)}{1/3 - 1/\mu_D}; \qquad t \in A_1 \text{ otherwise.} \tag{7}$$

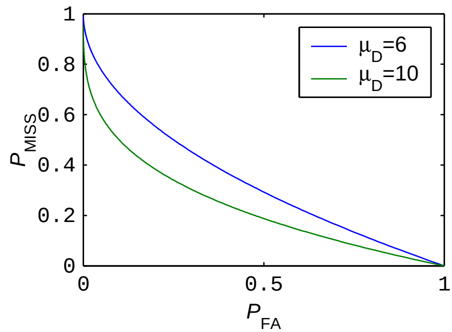
When $\mu_D = 6$ minutes, $t_{\mathrm{MAP}} = 6 \ln 8 = 12.48$ minutes. When $\mu_D = 10$ minutes, $t_{ML} = (30/7)\ln(40/3) = 11.1$ minutes.

(f) For a given threshold t_0, we learned in parts (a) and (b) that

$$P_{\text{FA}} = e^{-t_0/3}, \qquad\qquad P_{\text{MISS}} = 1 - e^{-t_0/\mu_D}. \tag{8}$$

The MATLAB program `rocvoicedataout` graphs both receiver operating curves. The program and the resulting ROC are shown here.

```
t=0:0.05:30;
PFA= exp(-t/3);
PMISS6= 1-exp(-t/6);
PMISS10=1-exp(-t/10);
plot(PFA,PMISS6,PFA,PMISS10);
legend('\mu_D=6','\mu_D=10');
xlabel('\itP_{\rmFA}');
ylabel('\itP_{\rmMISS}');
```



As one might expect, larger μ_D resulted in reduced P_{MISS} for the same P_{FA}.

Problem 11.2.4 Solution

(a) Given H_0, X is Gaussian $(0, 1)$. Given H_1, X is Gaussian $(4, 1)$. From Theorem 11.2, the MAP hypothesis test is

$$x \in A_0 \text{ if } \frac{f_{X|H_0}(x)}{f_{X|H_1}(x)} = \frac{e^{-x^2/2}}{e^{-(x-4)^2/2} \geq \frac{\text{P}[H_1]}{\text{P}[H_0]}}; \qquad x \in A_1 \text{ otherwise.} \tag{1}$$

Since a target is present with probability $\text{P}[H_1] = 0.01$, the MAP rule simplifies to

$$x \in A_0 \text{ if } x \leq x_{\text{MAP}}; \qquad x \in A_1 \text{ otherwise} \tag{2}$$

where

$$x_{\text{MAP}} = 2 - \frac{1}{4} \ln\left(\frac{\text{P}[H_1]}{\text{P}[H_0]}\right) = 3.15. \tag{3}$$

The false alarm and miss probabilities are

$$P_{\text{FA}} = \text{P}[X \geq x_{\text{MAP}}|H_0] = Q(x_{\text{MAP}}) = 8.16 \times 10^{-4}, \tag{4}$$

$$\begin{aligned} P_{\text{MISS}} &= \text{P}[X < x_{\text{MAP}}|H_1] \\ &= \Phi(x_{\text{MAP}} - 4) = 1 - \Phi(0.85) = 0.1977. \end{aligned} \tag{5}$$

The average cost of the MAP policy is

$$\begin{aligned} \text{E}[C_{\text{MAP}}] &= C_{10} P_{\text{FA}} \text{P}[H_0] + C_{01} P_{\text{MISS}} \text{P}[H_1] \\ &= (1)(8.16 \times 10^{-4})(0.99) + (10^4)(0.1977)(0.01) \\ &= 19.77. \end{aligned} \tag{6}$$

(b) The cost of a false alarm is $C_{10} = 1$ unit while the cost of a miss is $C_{01} = 10^4$ units. From Theorem 11.3, we see that the Minimum Cost test is the same as the MAP test except the $\text{P}[H_0]$ is replaced by $C_{10}\,\text{P}[H_0]$ and $\text{P}[H_1]$ is replaced by $C_{01}\,\text{P}[H_1]$. Thus, we see from the MAP test that the minimum cost test is

$$x \in A_0 \text{ if } x \leq x_{\text{MC}}; \qquad x \in A_1 \text{ otherwise.} \tag{7}$$

where

$$x_{\text{MC}} = 2 - \frac{1}{4} \ln\left(\frac{C_{01}\,\text{P}[H_1]}{C_{10}\,\text{P}[H_0]}\right) = 0.846. \tag{8}$$

The false alarm and miss probabilities are

$$P_{\text{FA}} = \text{P}[X \geq x_{\text{MC}}|H_0] = Q(x_{\text{MC}}) = 0.1987, \tag{9}$$

$$\begin{aligned} P_{\text{MISS}} &= \text{P}[X < x_{\text{MC}}|H_1] \\ &= \Phi(x_{\text{MC}} - 4) = 1 - \Phi(3.154) = 8.06 \times 10^{-4}. \end{aligned} \tag{10}$$

The average cost of the minimum cost policy is

$$\begin{aligned} \text{E}[C_{\text{MC}}] &= C_{10} P_{\text{FA}} \text{P}[H_0] + C_{01} P_{\text{MISS}} \text{P}[H_1] \\ &= (1)(0.1987)(0.99) + (10^4)(8.06 \times 10^{-4})(0.01) \\ &= 0.2773. \end{aligned} \tag{11}$$

Because the cost of a miss is so high, the minimum cost test greatly reduces the miss probability, resulting in a much lower average cost than the MAP test.