Probability and Stochastic Processes:
A Friendly Introduction for Electrical and Computer Engineers
Edition 3
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Yates and Goodman 3e Solution Set: 11.1.1, 11.1.4, 11.1.5, and 11.1.6

Problem 11.1.1 Solution

Assuming the coin is fair, we must choose a rejection region R such that $\alpha = \mathrm{P}[R] = 0.05$. We can choose a rejection region $R = \{L > r\}$. What remains is to choose r so that $\mathrm{P}[R] = 0.05$. Note that $L > l$ if we first observe l tails in a row. Under the hypothesis that the coin is fair, l tails in a row occurs with probability

$$\mathrm{P}[L > l] = (1/2)^l. \tag{1}$$

Thus, we need

$$\mathrm{P}[R] = \mathrm{P}[L > r] = 2^{-r} = 0.05. \tag{2}$$

Thus, $r = -\log_2(0.05) = \log_2(20) = 4.32$. In this case, we reject the hypothesis that the coin is fair if $L \geq 5$. The significance level of the test is $\alpha = \mathrm{P}[L > 4] = 2^{-4} = 0.0625$ which close to but not exactly 0.05.

The shortcoming of this test is that we always accept the hypothesis that the coin is fair whenever heads occurs on the first, second, third or fourth flip. If the coin was biased such that the probability of heads was much higher than 1/2, say 0.8 or 0.9, we would hardly ever reject the hypothesis that the coin is fair. In that sense, our test cannot identify that kind of biased coin.

Problem 11.1.4 Solution

(a) The rejection region is $R = \{T > t_0\}$. The duration of a voice call has exponential PDF

$$f_T(t) = \begin{cases} (1/3)e^{-t/3} & t \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The significance level of the test is

$$\alpha = P[T > t_0] = \int_{t_0}^{\infty} f_T(t)\, dt = e^{-t_0/3}. \tag{2}$$

(b) The significance level is $\alpha = 0.05$ if $t_0 = -3\ln\alpha = 8.99$ minutes.

Problem 11.1.5 Solution

In order to test just a small number of pacemakers, we test n pacemakers and we reject the null hypothesis if *any* pacemaker fails the test. Moreover, we choose the smallest n such that we meet the required significance level of the test.

The number of pacemakers that fail the test is X, a binomial $(n, q_0 = 10^{-4})$ random variable. The significance level of the test is

$$\alpha = P[X > 0] = 1 - P[X = 0] = 1 - (1 - q_0)^n. \tag{1}$$

For a significance level $\alpha = 0.01$, we have that

$$n = \frac{\ln(1-\alpha)}{\ln(1-q_0)} = 100.5. \tag{2}$$

Comment: For $\alpha = 0.01$, keep in mind that there is a one percent probability that a normal factory will fail the test. That is, a test failure is quite unlikely if the factory is operating normally.

Problem 11.1.6 Solution

(a) We wish to develop a hypothesis test of the form

$$P[|K - E[K]| > c] = 0.05. \tag{1}$$

to determine if the coin we've been flipping is indeed a fair one. We would like to find the value of c, which will determine the upper and lower limits on how many heads we can get away from the expected number out of 100 flips and still accept our hypothesis. Under our fair coin hypothesis, the expected number of heads, and the standard deviation of the process are

$$\mathrm{E}[K] = 50, \qquad \sigma_K = \sqrt{100 \cdot 1/2 \cdot 1/2} = 5. \tag{2}$$

Now in order to find c we make use of the central limit theorem and divide the above inequality through by σ_K to arrive at

$$\mathrm{P}\left[\frac{|K - \mathrm{E}[K]|}{\sigma_K} > \frac{c}{\sigma_K}\right] = 0.05. \tag{3}$$

Taking the complement, we get

$$\mathrm{P}\left[-\frac{c}{\sigma_K} \leq \frac{K - \mathrm{E}[K]}{\sigma_K} \leq \frac{c}{\sigma_K}\right] = 0.95. \tag{4}$$

Using the Central Limit Theorem we can write

$$\Phi\left(\frac{c}{\sigma_K}\right) - \Phi\left(\frac{-c}{\sigma_K}\right) = 2\Phi\left(\frac{c}{\sigma_K}\right) - 1 = 0.95. \tag{5}$$

This implies $\Phi(c/\sigma_K) = 0.975$ or $c/5 = 1.96$. That is, $c = 9.8$ flips. So we see that if we observe more then $50 + 10 = 60$ or less then $50 - 10 = 40$ heads, then with significance level $\alpha \approx 0.05$ we should reject the hypothesis that the coin is fair.

(b) Now we wish to develop a test of the form

$$\mathrm{P}[K > c] = 0.01. \tag{6}$$

Thus we need to find the value of c that makes the above probability true. This value will tell us that if we observe more than c heads, then with significance level $\alpha = 0.01$, we should reject the hypothesis that the coin is fair. To find this value of c we look to evaluate the CDF

$$F_K(k) = \sum_{i=0}^{k} \binom{100}{i} (1/2)^{100}. \tag{7}$$

Computation reveals that $c \approx 62$ flips. So if we observe 62 or greater heads, then with a significance level of 0.01 we should reject the fair coin hypothesis.

Another way to obtain this result is to use a Central Limit Theorem approximation. First, we express our rejection region in terms of a zero mean, unit variance random variable.

$$\begin{aligned}
\mathrm{P}\left[K > c\right] &= 1 - \mathrm{P}\left[K \leq c\right] \\
&= 1 - \mathrm{P}\left[\frac{K - \mathrm{E}\left[K\right]}{\sigma_K} \leq \frac{c - \mathrm{E}\left[K\right]}{\sigma_K}\right] = 0.01. \qquad (8)
\end{aligned}$$

Since $\mathrm{E}[K] = 50$ and $\sigma_K = 5$, the CLT approximation is

$$\mathrm{P}\left[K > c\right] \approx 1 - \Phi\left(\frac{c - 50}{5}\right) = 0.01. \qquad (9)$$

From Table 4.2, we have $(c - 50)/5 = 2.35$ or $c = 61.75$. Once again, we see that we reject the hypothesis if we observe 62 or more heads.