Probability and Stochastic Processes:
A Friendly Introduction for Electrical and Computer Engineers
Edition 3

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Yates and Goodman 3e Solution Set: 10.1.1, 10.1.3, 10.1.4, 10.2.1, 10.2.3, 10.3.1, and 10.3.2

Problem 10.1.1 Solution

Recall that $X_1, X_2 \ldots X_n$ are independent exponential random variables with mean value $\mu_X = 5$ so that for $x \geq 0$, $F_X(x) = 1 - e^{-x/5}$.

(a) Using Theorem 10.1, $\sigma^2_{M_n(x)} = \sigma^2_X / n$. Realizing that $\sigma^2_X = 25$, we obtain

$$\operatorname{Var}[M_9(X)] = \frac{\sigma^2_X}{9} = \frac{25}{9}. \tag{1}$$

(b)

$$\begin{aligned} \mathrm{P}[X_1 \geq 7] &= 1 - \mathrm{P}[X_1 \leq 7] \\ &= 1 - F_X(7) \\ &= 1 - (1 - e^{-7/5}) = e^{-7/5} \approx 0.247. \end{aligned} \tag{2}$$

(c) First we express $\mathrm{P}[M_9(X) > 7]$ in terms of $X_1, \ldots, X_9$.

$$\begin{aligned} \mathrm{P}[M_9(X) > 7] &= 1 - \mathrm{P}[M_9(X) \leq 7] \\ &= 1 - \mathrm{P}[(X_1 + \ldots + X_9) \leq 63]. \end{aligned} \tag{3}$$

Now the probability that $M_9(X) > 7$ can be approximated using the Central Limit Theorem (CLT).

$$\begin{aligned} \mathrm{P}[M_9(X) > 7] &= 1 - \mathrm{P}[(X_1 + \ldots + X_9) \leq 63] \\ &\approx 1 - \Phi\left(\frac{63 - 9\mu_X}{\sqrt{9}\sigma_X}\right) \\ &= 1 - \Phi(6/5). \end{aligned} \tag{4}$$

Consulting with Table 4.2 yields $\mathrm{P}[M_9(X) > 7] \approx 0.1151$.

Problem 10.1.3 Solution

This problem is in the wrong section since the *standard error* isn't defined until Section 10.4. However is we peek ahead to this section, the problem isn't very hard. Given the sample mean estimate $M_n(X)$, the standard error is defined as the standard deviation $e_n = \sqrt{\mathrm{Var}[M_n(X)]}$. In our problem, we use samples X_i to generate $Y_i = X_i^2$. For the sample mean $M_n(Y)$, we need to find the standard error

$$e_n = \sqrt{\mathrm{Var}[M_n(Y)]} = \sqrt{\frac{\mathrm{Var}[Y]}{n}}. \tag{1}$$

Since X is a uniform $(0,1)$ random variable,

$$\mathrm{E}[Y] = \mathrm{E}\left[X^2\right] = \int_0^1 x^2\,dx = 1/3, \tag{2}$$

$$\mathrm{E}\left[Y^2\right] = \mathrm{E}\left[X^4\right] = \int_0^1 x^4\,dx = 1/5. \tag{3}$$

Thus $\mathrm{Var}[Y] = 1/5 - (1/3)^2 = 4/45$ and the sample mean $M_n(Y)$ has standard error

$$e_n = \sqrt{\frac{4}{45n}}. \tag{4}$$

Problem 10.1.4 Solution

(a) Since $Y_n = X_{2n-1} + (-X_{2n})$, Theorem 9.1 says that the expected value of the difference is

$$\mathrm{E}[Y] = \mathrm{E}[X_{2n-1}] + \mathrm{E}[-X_{2n}] = \mathrm{E}[X] - \mathrm{E}[X] = 0. \tag{1}$$

By Theorem 9.2, the variance of the difference between X_{2n-1} and X_{2n} is

$$\mathrm{Var}[Y_n] = \mathrm{Var}[X_{2n-1}] + \mathrm{Var}[-X_{2n}] = 2\,\mathrm{Var}[X]. \tag{2}$$

(b) Each Y_n is the difference of two samples of X that are independent of the samples used by any other Y_m. Thus $Y_1, Y_2, \ldots$ is an iid random sequence. By Theorem 10.1, the mean and variance of $M_n(Y)$ are

$$\mathrm{E}\left[M_n(Y)\right] = \mathrm{E}\left[Y_n\right] = 0, \tag{3}$$
$$\mathrm{Var}[M_n(Y)] = \frac{\mathrm{Var}[Y_n]}{n} = \frac{2\,\mathrm{Var}[X]}{n}. \tag{4}$$

Problem 10.2.1 Solution

If the average weight of a Maine black bear is 500 pounds with standard deviation equal to 100 pounds, we can use the Chebyshev inequality to upper bound the probability that a randomly chosen bear will be more then 200 pounds away from the average.

$$\mathrm{P}\left[|W - \mathrm{E}\left[W\right]| \geq 200\right] \leq \frac{\mathrm{Var}[W]}{200^2} \leq \frac{100^2}{200^2} = 0.25. \tag{1}$$

Problem 10.2.3 Solution

The arrival time of the third elevator is $W = X_1 + X_2 + X_3$. Since each X_i is uniform $(0, 30)$, $\mathrm{E}[X_i] = 15$ and $\mathrm{Var}[X_i] = (30-0)^2/12 = 75$. Thus $\mathrm{E}[W] = 3\,\mathrm{E}[X_i] = 45$, and $\mathrm{Var}[W] = 3\,\mathrm{Var}[X_i] = 225$.

(a) By the Markov inequality,

$$\mathrm{P}\left[W > 75\right] \leq \frac{\mathrm{E}\left[W\right]}{75} = \frac{45}{75} = \frac{3}{5}. \tag{1}$$

(b) By the Chebyshev inequality,

$$\begin{aligned}\mathrm{P}\left[W > 75\right] &= \mathrm{P}\left[W - \mathrm{E}\left[W\right] > 30\right] \\ &\leq \mathrm{P}\left[|W - \mathrm{E}\left[W\right]| > 30\right] \\ &\leq \frac{\mathrm{Var}\left[W\right]}{30^2} = \frac{1}{4}.\end{aligned} \tag{2}$$

Problem 10.3.1 Solution

$X_1, X_2, \ldots$ are iid random variables each with mean 75 and standard deviation 15.

(a) We would like to find the value of n such that

$$P[74 \leq M_n(X) \leq 76] = 0.99. \tag{1}$$

When we know only the mean and variance of X_i, our only real tool is the Chebyshev inequality which says that

$$\begin{aligned} P[74 \leq M_n(X) \leq 76] &= 1 - P[|M_n(X) - E[X]| \geq 1] \\ &\geq 1 - \frac{\text{Var}[X]}{n} = 1 - \frac{225}{n} \geq 0.99. \end{aligned} \tag{2}$$

This yields $n \geq 22{,}500$.

(b) If each X_i is a Gaussian, the sample mean, $M_n(X)$ will also be Gaussian with mean and variance

$$E[M_{n'}(X)] = E[X] = 75, \tag{3}$$

$$\text{Var}[M_{n'}(X)] = \text{Var}[X]/n' = 225/n' \tag{4}$$

In this case,

$$\begin{aligned} P[74 \leq M_{n'}(X) \leq 76] &= \Phi\left(\frac{76-\mu}{\sigma}\right) - \Phi\left(\frac{74-\mu}{\sigma}\right) \\ &= \Phi(\sqrt{n'}/15) - \Phi(-\sqrt{n'}/15) \\ &= 2\Phi(\sqrt{n'}/15) - 1 = 0.99. \end{aligned} \tag{5}$$

Thus, $n' = 1{,}521$.

Since even under the Gaussian assumption, the number of samples n' is so large that even if the X_i are not Gaussian, the sample mean may be approximated by a Gaussian. Hence, about 1500 samples probably is about right. However, in the absence of any information about the PDF of X_i beyond the mean and variance, we cannot make any guarantees stronger than that given by the Chebyshev inequality.

Problem 10.3.2 Solution

(a) Since X_A is a Bernoulli ($p = \mathrm{P}[A]$) random variable,

$$\mathrm{E}[X_A] = \mathrm{P}[A] = 0.8, \qquad \mathrm{Var}[X_A] = \mathrm{P}[A](1 - \mathrm{P}[A]) = 0.16. \tag{1}$$

(b) Let $X_{A,i}$ to denote X_A on the ith trial. Since

$$\hat{P}_n(A) = M_n(X_A) = \frac{1}{n}\sum_{i=1}^{n} X_{A,i}, \tag{2}$$

is a sum of n independent random variables,

$$\mathrm{Var}[\hat{P}_n(A)] = \frac{1}{n^2}\sum_{i=1}^{n} \mathrm{Var}[X_{A,i}] = \frac{\mathrm{P}[A](1 - \mathrm{P}[A])}{n}. \tag{3}$$

(c) Since $\hat{P}_{100}(A) = M_{100}(X_A)$, we can use Theorem 10.5(b) to write

$$\begin{aligned} \mathrm{P}\left[\left|\hat{P}_{100}(A) - \mathrm{P}[A]\right| < c\right] &\geq 1 - \frac{\mathrm{Var}[X_A]}{100c^2} \\ &= 1 - \frac{0.16}{100c^2} = 1 - \alpha. \end{aligned} \tag{4}$$

For $c = 0.1$, $\alpha = 0.16/[100(0.1)^2] = 0.16$. Thus, with 100 samples, our confidence coefficient is $1 - \alpha = 0.84$.

(d) In this case, the number of samples n is unknown. Once again, we use Theorem 10.5(b) to write

$$\begin{aligned} \mathrm{P}\left[\left|\hat{P}_n(A) - \mathrm{P}[A]\right| < c\right] &\geq 1 - \frac{\mathrm{Var}[X_A]}{nc^2} \\ &= 1 - \frac{0.16}{nc^2} = 1 - \alpha. \end{aligned} \tag{5}$$

For $c = 0.1$, we have confidence coefficient $1 - \alpha = 0.95$ if $\alpha = 0.16/[n(0.1)^2] = 0.05$, or $n = 320$.