Probability and Stochastic Processes:
A Friendly Introduction for Electrical and Computer Engineers
Edition 3
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Yates and Goodman 3e Solution Set: 1.6.6, 1.6.8, 1.6.10, 2.1.4, 2.1.5, 2.1.9, 2.2.5, 2.2.6, 2.2.7, 2.2.12, 2.3.1, 2.3.3, 2.4.1, 2.4.2, and 2.4.3

Problem 1.6.6 Solution

(a) Since C and D are independent,

$$P[C \cap D] = P[C] P[D] = 15/64. \tag{1}$$

The next few items are a little trickier. From Venn diagrams, we see

$$P[C \cap D^c] = P[C] - P[C \cap D] = 5/8 - 15/64 = 25/64. \tag{2}$$

It follows that

$$P[C \cup D^c] = P[C] + P[D^c] - P[C \cap D^c] \tag{3}$$
$$= 5/8 + (1 - 3/8) - 25/64 = 55/64. \tag{4}$$

Using DeMorgan's law, we have

$$P[C^c \cap D^c] = P[(C \cup D)^c] = 1 - P[C \cup D] = 15/64. \tag{5}$$

(b) Since $P[C^c D^c] = P[C^c] P[D^c]$, C^c and D^c are independent.

Problem 1.6.8 Solution

(a) Since C and D are independent $P[CD] = P[C] P[D]$. So

$$P[D] = \frac{P[CD]}{P[C]} = \frac{1/3}{1/2} = 2/3. \tag{1}$$

In addition, $P[C \cap D^c] = P[C] - P[C \cap D] = 1/2 - 1/3 = 1/6$. To find $P[C^c \cap D^c]$, we first observe that

$$P[C \cup D] = P[C] + P[D] - P[C \cap D] = 1/2 + 2/3 - 1/3 = 5/6. \quad (2)$$

By De Morgan's Law, $C^c \cap D^c = (C \cup D)^c$. This implies

$$P[C^c \cap D^c] = P[(C \cup D)^c] = 1 - P[C \cup D] = 1/6. \quad (3)$$

Note that a second way to find $P[C^c \cap D^c]$ is to use the fact that if C and D are independent, then C^c and D^c are independent. Thus

$$P[C^c \cap D^c] = P[C^c] P[D^c] = (1 - P[C])(1 - P[D]) = 1/6. \quad (4)$$

Finally, since C and D are independent events, $P[C|D] = P[C] = 1/2$.

(b) Note that we found $P[C \cup D] = 5/6$. We can also use the earlier results to show

$$P[C \cup D^c] = P[C] + P[D] - P[C \cap D^c] \quad (5)$$
$$= 1/2 + (1 - 2/3) - 1/6 = 2/3. \quad (6)$$

(c) By Definition 1.6, events C and D^c are independent because

$$P[C \cap D^c] = 1/6 = (1/2)(1/3) = P[C] P[D^c]. \quad (7)$$

Problem 1.6.10 Solution

There are 16 distinct equally likely outcomes for the second generation of pea plants based on a first generation of $\{rwyg, rwgy, wryg, wrgy\}$. These are:

$$\begin{array}{cccc} rryy & rryg & rrgy & rrgg \\ rwyy & rwyg & rwgy & rwgg \\ wryy & wryg & wrgy & wrgg \\ wwyy & wwyg & wwgy & wwgg \end{array}$$

A plant has yellow seeds, that is event Y occurs, if a plant has at least one dominant y gene. Except for the four outcomes with a pair of recessive g genes, the remaining 12 outcomes have yellow seeds. From the above, we see that

$$P[Y] = 12/16 = 3/4 \quad (1)$$

and

$$\mathrm{P}[R] = 12/16 = 3/4. \tag{2}$$

To find the conditional probabilities $\mathrm{P}[R|Y]$ and $\mathrm{P}[Y|R]$, we first must find $\mathrm{P}[RY]$. Note that RY, the event that a plant has rounded yellow seeds, is the set of outcomes

$$RY = \{rryy, rryg, rrgy, rwyy, rwyg, rwgy, wryy, wryg, wrgy\}. \tag{3}$$

Since $\mathrm{P}[RY] = 9/16$,

$$\mathrm{P}[Y|R] = \frac{\mathrm{P}[RY]}{\mathrm{P}[R]} = \frac{9/16}{3/4} = 3/4 \tag{4}$$

and

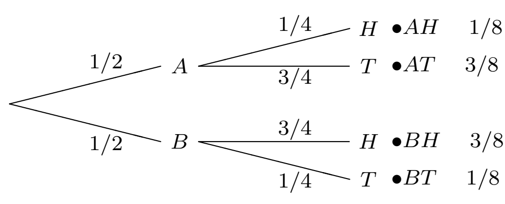
$$\mathrm{P}[R|Y] = \frac{\mathrm{P}[RY]}{\mathrm{P}[Y]} = \frac{9/16}{3/4} = 3/4. \tag{5}$$

Thus $\mathrm{P}[R|Y] = \mathrm{P}[R]$ and $\mathrm{P}[Y|R] = \mathrm{P}[Y]$ and R and Y are independent events. There are four visibly different pea plants, corresponding to whether the peas are round (R) or not (R^c), or yellow (Y) or not (Y^c). These four visible events have probabilities

$$\mathrm{P}[RY] = 9/16 \qquad \mathrm{P}[RY^c] = 3/16, \tag{6}$$
$$\mathrm{P}[R^cY] = 3/16 \qquad \mathrm{P}[R^cY^c] = 1/16. \tag{7}$$

Problem 2.1.4 Solution

The tree for this experiment is



The probability that you guess correctly is

$$\mathrm{P}[C] = \mathrm{P}[AT] + \mathrm{P}[BH] = 3/8 + 3/8 = 3/4. \tag{1}$$

Problem 2.1.5 Solution

The $P[-|H]$ is the probability that a person who has HIV tests negative for the disease. This is referred to as a false-negative result. The case where a person who does not have HIV but tests positive for the disease, is called a false-positive result and has probability $P[+|H^c]$. Since the test is correct 99% of the time,

$$P[-|H] = P[+|H^c] = 0.01. \tag{1}$$

Now the probability that a person who has tested positive for HIV actually has the disease is

$$P[H|+] = \frac{P[+,H]}{P[+]} = \frac{P[+,H]}{P[+,H] + P[+,H^c]}. \tag{2}$$

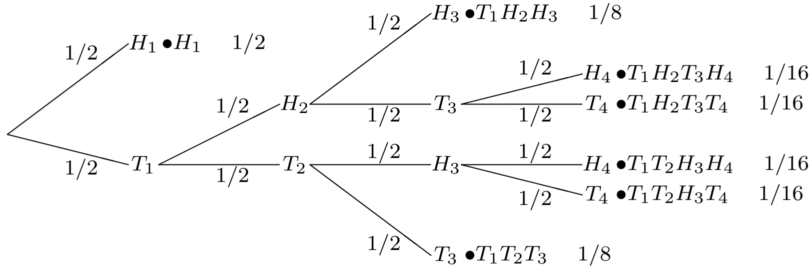
We can use Bayes' formula to evaluate these joint probabilities.

$$\begin{aligned} P[H|+] &= \frac{P[+|H] P[H]}{P[+|H] P[H] + P[+|H^c] P[H^c]} \\ &= \frac{(0.99)(0.0002)}{(0.99)(0.0002) + (0.01)(0.9998)} \\ &= 0.0194. \end{aligned} \tag{3}$$

Thus, even though the test is correct 99% of the time, the probability that a random person who tests positive actually has HIV is less than 0.02. The reason this probability is so low is that the a priori probability that a person has HIV is very small.

Problem 2.1.9 Solution

(a) The primary difficulty in this problem is translating the words into the correct tree diagram. The tree for this problem is shown below.



(b) From the tree,

$$\begin{aligned} P[H_3] &= P[T_1H_2H_3] + P[T_1T_2H_3H_4] + P[T_1T_2H_3H_4] \\ &= 1/8 + 1/16 + 1/16 = 1/4. \end{aligned} \tag{1}$$

Similarly,

$$\begin{aligned} P[T_3] &= P[T_1H_2T_3H_4] + P[T_1H_2T_3T_4] + P[T_1T_2T_3] \\ &= 1/8 + 1/16 + 1/16 = 1/4. \end{aligned} \tag{2}$$

(c) The event that Dagwood must diet is

$$D = (T_1H_2T_3T_4) \cup (T_1T_2H_3T_4) \cup (T_1T_2T_3). \tag{3}$$

The probability that Dagwood must diet is

$$\begin{aligned} P[D] &= P[T_1H_2T_3T_4] + P[T_1T_2H_3T_4] + P[T_1T_2T_3] \\ &= 1/16 + 1/16 + 1/8 = 1/4. \end{aligned} \tag{4}$$

The conditional probability of heads on flip 1 given that Dagwood must diet is

$$P[H_1|D] = \frac{P[H_1D]}{P[D]} = 0. \tag{5}$$

Remember, if there was heads on flip 1, then Dagwood always postpones his diet.

(d) From part (b), we found that $P[H_3] = 1/4$. To check independence, we calculate

$$\begin{aligned} P[H_2] &= P[T_1 H_2 H_3] + P[T_1 H_2 T_3] + P[T_1 H_2 T_4 T_4] = 1/4 \\ P[H_2 H_3] &= P[T_1 H_2 H_3] = 1/8. \end{aligned} \tag{6}$$

Now we find that

$$P[H_2 H_3] = 1/8 \neq P[H_2] P[H_3]. \tag{7}$$

Hence, H_2 and H_3 are dependent events. In fact, $P[H_3|H_2] = 1/2$ while $P[H_3] = 1/4$. The reason for the dependence is that given H_2 occurred, then we know there will be a third flip which may result in H_3. That is, knowledge of H_2 tells us that the experiment didn't end after the first flip.

Problem 2.2.5 Solution

Since there are $H = \binom{52}{7}$ equiprobable seven-card hands, each probability is just the number of hands of each type divided by H.

(a) Since there are 26 red cards, there are $\binom{26}{7}$ seven-card hands with all red cards. The probability of a seven-card hand of all red cards is

$$P[R_7] = \frac{\binom{26}{7}}{\binom{52}{7}} = \frac{26!\,45!}{52!\,19!} = 0.0049. \tag{1}$$

(b) There are 12 face cards in a 52 card deck and there are $\binom{12}{7}$ seven card hands with all face cards. The probability of drawing only face cards is

$$P[F] = \frac{\binom{12}{7}}{\binom{52}{7}} = \frac{12!\,45!}{5!\,52!} = 5.92 \times 10^{-6}. \tag{2}$$

(c) There are 6 red face cards (J, Q, K of diamonds and hearts) in a 52 card deck. Thus it is impossible to get a seven-card hand of red face cards: $P[R_7 F] = 0$.

Problem 2.2.6 Solution

There are $H_5 = \binom{52}{5}$ equally likely five-card hands. Dividing the number of hands of a particular type by H will yield the probability of a hand of that type.

(a) There are $\binom{26}{5}$ five-card hands of all red cards. Thus the probability getting a five-card hand of all red cards is

$$P[R_5] = \frac{\binom{26}{5}}{\binom{52}{5}} = \frac{26!\,47!}{21!\,52!} = 0.0253. \tag{1}$$

Note that this can be rewritten as

$$P[R_5] = \frac{26}{52}\frac{25}{51}\frac{24}{50}\frac{23}{49}\frac{22}{48},$$

which shows the successive probabilities of receiving a red card.

(b) The following sequence of subexperiments will generate all possible "full house"

1. Choose a kind for three-of-a-kind.
2. Choose a kind for two-of-a-kind.
3. Choose three of the four cards of the three-of-a-kind kind.
4. Choose two of the four cards of the two-of-a-kind kind.

The number of ways of performing subexperiment i is

$$n_1 = \binom{13}{1}, \qquad n_2 = \binom{12}{1}, \qquad n_3 = \binom{4}{3}, \qquad n_4 = \binom{4}{2}. \tag{2}$$

Note that $n_2 = \binom{12}{1}$ because after choosing a three-of-a-kind, there are twelove kinds left from which to choose two-of-a-kind. is

The probability of a full house is

$$P[\text{full house}] = \frac{n_1 n_2 n_3 n_4}{\binom{52}{5}} = \frac{3,744}{2,598,960} = 0.0014. \tag{3}$$

Problem 2.2.7 Solution

There are $2^5 = 32$ different binary codes with 5 bits. The number of codes with exactly 3 zeros equals the number of ways of choosing the bits in which those zeros occur. Therefore there are $\binom{5}{3} = 10$ codes with exactly 3 zeros.

Problem 2.2.12 Solution

(a) We can find the number of valid starting lineups by noticing that the swingman presents three situations: (1) the swingman plays guard, (2) the swingman plays forward, and (3) the swingman doesn't play. The first situation is when the swingman can be chosen to play the guard position, and the second where the swingman can only be chosen to play the forward position. Let N_i denote the number of lineups corresponding to case i. Then we can write the total number of lineups as $N_1 + N_2 + N_3$. In the first situation, we have to choose 1 out of 3 centers, 2 out of 4 forwards, and 1 out of 4 guards so that

$$N_1 = \binom{3}{1}\binom{4}{2}\binom{4}{1} = 72. \tag{1}$$

In the second case, we need to choose 1 out of 3 centers, 1 out of 4 forwards and 2 out of 4 guards, yielding

$$N_2 = \binom{3}{1}\binom{4}{1}\binom{4}{2} = 72. \tag{2}$$

Finally, with the swingman on the bench, we choose 1 out of 3 centers, 2 out of 4 forward, and 2 out of four guards. This implies

$$N_3 = \binom{3}{1}\binom{4}{2}\binom{4}{2} = 108, \tag{3}$$

and the total number of lineups is $N_1 + N_2 + N_3 = 252$.

Problem 2.3.1 Solution

(a) Since the probability of a zero is 0.8, we can express the probability of the code word 00111 as 2 occurrences of a 0 and three occurrences of a 1. Therefore

$$\mathrm{P}[00111] = (0.8)^2(0.2)^3 = 0.00512. \tag{1}$$

(b) The probability that a code word has exactly three 1's is

$$\mathrm{P}[\text{three 1's}] = \binom{5}{3}(0.8)^2(0.2)^3 = 0.0512. \tag{2}$$

Problem 2.3.3 Solution

We know that the probability of a green and red light is 7/16, and that of a yellow light is 1/8. Since there are always 5 lights, G, Y, and R obey the multinomial probability law:

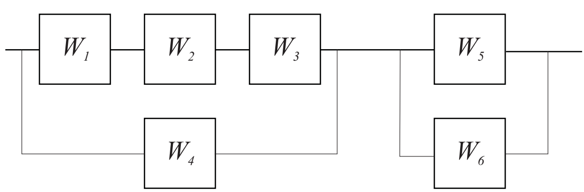
$$\mathrm{P}[G = 2, Y = 1, R = 2] = \frac{5!}{2!1!2!}\left(\frac{7}{16}\right)^2\left(\frac{1}{8}\right)\left(\frac{7}{16}\right)^2. \tag{1}$$

The probability that the number of green lights equals the number of red lights

$$\begin{aligned}\mathrm{P}[G = R] = \mathrm{P}[G = 1, R = 1, Y = 3] + \mathrm{P}[G = 2, R = 2, Y = 1]&\\ + \mathrm{P}[G = 0, R = 0, Y = 5]&\\ = \frac{5!}{1!1!3!}\left(\frac{7}{16}\right)\left(\frac{7}{16}\right)\left(\frac{1}{8}\right)^3 + \frac{5!}{2!1!2!}\left(\frac{7}{16}\right)^2\left(\frac{7}{16}\right)^2\left(\frac{1}{8}\right)&\\ + \frac{5!}{0!0!5!}\left(\frac{1}{8}\right)^5&\\ \approx 0.1449.&\end{aligned} \tag{2}$$

Problem 2.4.1 Solution

From the problem statement, we can conclude that the device components are configured in the following way.

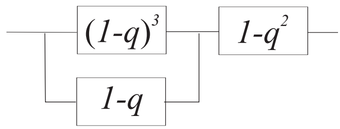


To find the probability that the device works, we replace series devices 1, 2, and 3, and parallel devices 5 and 6 each with a single device labeled with the probability that it works. In particular,

$$P[W_1W_2W_3] = (1-q)^3, \tag{1}$$

$$P[W_5 \cup W_6] = 1 - P[W_5^c W_6^c] = 1 - q^2. \tag{2}$$

This yields a composite device of the form



The probability $P[W']$ that the two devices in parallel work is 1 minus the probability that neither works:

$$P[W'] = 1 - q(1-(1-q)^3). \tag{3}$$

Finally, for the device to work, both composite device in series must work. Thus, the probability the device works is

$$P[W] = [1 - q(1-(1-q)^3)][1-q^2]. \tag{4}$$

Problem 2.4.2 Solution

Suppose that the transmitted bit was a 1. We can view each repeated transmission as an independent trial. We call each repeated bit the receiver decodes as 1 a success. Using $S_{k,5}$ to denote the event of k successes in the five trials, then the probability k 1's are decoded at the receiver is

$$P[S_{k,5}] = \binom{5}{k} p^k (1-p)^{5-k}, \quad k = 0, 1, \ldots, 5. \tag{1}$$

The probability a bit is decoded correctly is

$$P[C] = P[S_{5,5}] + P[S_{4,5}] = p^5 + 5p^4(1-p) = 0.91854. \tag{2}$$

The probability a deletion occurs is

$$P[D] = P[S_{3,5}] + P[S_{2,5}] = 10p^3(1-p)^2 + 10p^2(1-p)^3 = 0.081. \quad (3)$$

The probability of an error is

$$P[E] = P[S_{1,5}] + P[S_{0,5}] = 5p(1-p)^4 + (1-p)^5 = 0.00046. \quad (4)$$

Note that if a 0 is transmitted, then 0 is sent five times and we call decoding a 0 a success. You should convince yourself that this a symmetric situation with the same deletion and error probabilities. Introducing deletions reduces the probability of an error by roughly a factor of 20. However, the probability of successfull decoding is also reduced.

Problem 2.4.3 Solution

Note that each digit 0 through 9 is mapped to the 4 bit binary representation of the digit. That is, 0 corresponds to 0000, 1 to 0001, up to 9 which corresponds to 1001. Of course, the 4 bit binary numbers corresponding to numbers 10 through 15 go unused, however this is unimportant to our problem. the 10 digit number results in the transmission of 40 bits. For each bit, an independent trial determines whether the bit was correct, a deletion, or an error. In Problem 2.4.2, we found the probabilities of these events to be

$$P[C] = \gamma = 0.91854, \qquad P[D] = \delta = 0.081, \qquad P[E] = \epsilon = 0.00046. \quad (1)$$

Since each of the 40 bit transmissions is an independent trial, the joint probability of c correct bits, d deletions, and e erasures has the multinomial probability

$$P[C = c, D = d, E = d] = \begin{cases} \frac{40!}{c!d!e!}\gamma^c\delta^d\epsilon^e & c+d+e = 40; c,d,e \geq 0, \\ 0 & \text{otherwise.} \end{cases} \quad (2)$$