De Quantitate

G.W. Leibniz, 1680-1682
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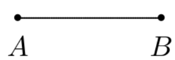
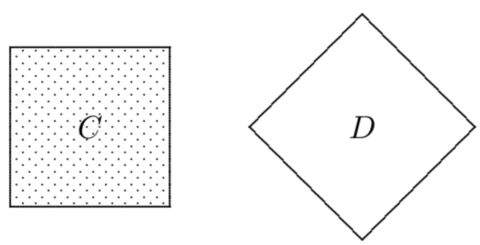
[Manuscript]
[Typescript]

Determiners are things that accord simultaneously with only one thing. As two extremes A, B accord with just one straight line.



[Fig. 1]

Coincidents are things that are completely the same and differ only by denomination, as the path from A to B and the path from B to A.

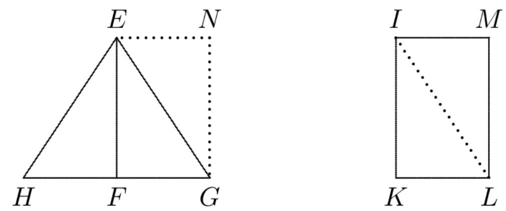


[Fig. 2]

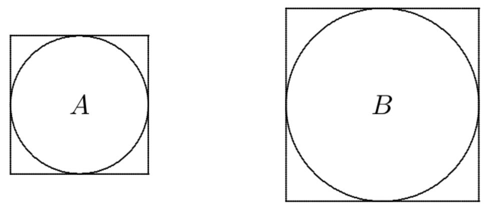
Congruents are things that, if they are distinct, they cannot be distinguished except with respect to something external, as [for instance] the squares C and D, since of course they are in different places or situations at the same time, or because one C is in a gold material, the other D in silver. Likewise a gold pound and a silver pound are congruent, [and] the days today and yesterday. Any point whatever is congruent to any other, as also any instant to any instant.

Equals are things that are either congruent (for example, triangles EFH, EFG, IKL, LMI, GNE, and likewise rectangles $EFGN$ et $IKLM$), or can be rendered congruent by a transformation (as triangle HEG to rectangle $IKLM$, since if part of HEG, namely EFH, is transposed to GNE, which can be done because they are congruent, then HEG is transformed to $FGEN$, congruent to $IKLM$; and so HEG and $IKLM$ are called equals). Therefore *Equals* can be defined as those which can be resolved into separate parts, each congruent individually to [a part] of the other.

Similars are those in which nothing may be discovered by which they can be distinguished, if they are considered in themselves individually, as two spheres or circles (or two



[Fig. 3]



[Fig. 4]

cubes or two perfect squares) A and B. If the eye only, without any other member, is imagined to be now inside sphere A, now inside sphere B, it will not be able to distinguish them; but it can if both are seen simultaneously, or if another limb of the body or another measure were brought inside with it, which is applied now to one, now to the other. Thus to distinguish similars, we need the co-presence either of them with each other, or of a third with each in succession. [But in dissimilars, something noted in one by the proportion of its parts, which is not noted in the other, suffices for distinguishing them individually; of which more later.]

[Fig. 5]

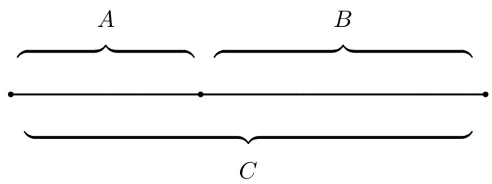
Homogeneous things are those which either are similar, or can be converted to similars by a transformation. Two lines are homogeneous, since they are similar; but a line and a circular arc are homogeneous, since the circle can be stretched into a line.

If there are multiple things such as A and B, and one thing C, and they are coming together in something, and in these a homogeneous thing is-in A and B in common, but all homogenea in them are common to C; then the several things are called *integral parts* and the one thing is called the *whole*.

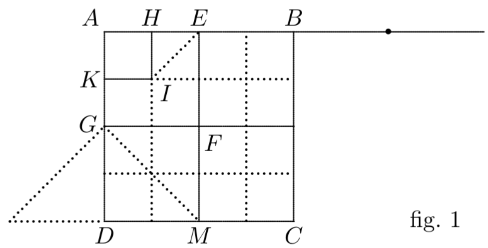
We could also define *Homogeneous* things as those which come together in something, in which also others come together that can be assumed indefinitely in either one of them.

The *Lesser* is what is equal to a part of the other (*the Greater*). *Quantity* is that which accords to a thing insofar as it has all its parts; or i.e. on account of which it may be said to be equal to, greater than, or lesser than another thing (any homogeneous thing), or i.e. may be compared to it.

The quantity of an object, for example of the area $ABCD$ in Fig. 7], is expressed by a number, for example four, with another thing such as the square foot $AEFG$ taken to be a primary Measure or real Unity. Indeed $ABCD$ is four square feet. But if another unity $AHIK$ is assumed, which is the square of a half foot AH, the quantity of the area $ABCD$ will be 16. Thus, according to the unity assumed, various numbers will result for the same



[Fig. 6]



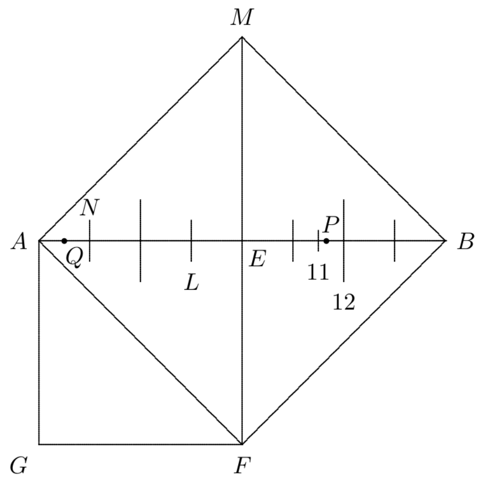
[Fig. 7]

quantity. And hence quantity is not a definite number, but the material of a number or an indefinite number, to be defined when a fixed measure is assumed. Quantities, therefore, are expressed by definite numbers, like 1, 2, or indefinite, by letters or other characters a, b, $\odot$, ☽.

A *number* is something homogeneous with Unity and thus can be compared with unity and added to and subtracted from it. It is either an aggregate of unities, which is called an *integer*, such as 2 (or $1+1$), and likewise 3, 4 (whether $2+1$ or $1+1+1$), or else an aggregate of some parts of unity, which is called a *Fraction*; if a unity, for example a foot AB, is divided in four parts, then something such as the line BH, which has three quarters of a foot, or three of $\frac{1}{4}$, will thus be expressed $\frac{3}{4}$; and sometimes a fraction may be reduced to an integer, as AB or 2 is four of AH, or 4 halves, or $\frac{4}{2}$; or, finally, a number is determined in some other way by relation to unity, and in fact these relations could be infinite, but the most customary are by radicals. Indeed, for the number 4 (for the square $ABCD$), suppose its square root is sought (or the side AB); it is the number which, multiplied by itself, makes 4; the number is 2, and so since $2 \cdot 2$ or aa is 4, $\sqrt{4}$ ($\sqrt{aa}$) is 2 (a). And in this case the radical can be reduced to a common number or *rational*.

But sometimes this reduction does not succeed. For example, if a number is sought which multiplied by itself would make 2, it certainly isn't an integer (for otherwise, since it is necessarily less than 2, it would be unity, but unity multiplied by itself makes 1), and neither is it a fraction, since every fraction multiplied by itself produces another fraction, as $\frac{3}{2}$ produces $\frac{9}{4}$ or $2+\frac{1}{4}$; and thus the number is nothing but *irrational*, as they say, or rather ineffable, alogos, a *surd*, which is written another way as $\sqrt{2}$, or $\sqrt{q}\,2.$, or $\sqrt[2]{2}$, that is, the quadratic root of 2; meaning: having set this number to be y, then its square yy or y^2 is 2. And so that it's clear that this number is in the nature of things: let the diagonal AF (Fig. 8) be drawn in the square $AEFG$. If AG is 1, one foot of course, whose square is 1 (the space $AEFG$ of course, or one foot square), then AF, which we call y, will be $\sqrt{2}$; for its square yy, $AFBM$, is 2 (of course, double a foot squared); the square $AFBM$ is in fact double of the square $AEFG$, for its half, triangle AFB, is equal to the whole $AEFG$. Since, therefore, we define number as something homogeneous to unity, certainly there should be some number having the ratio to unity that the line AF has to the line AG, or having set AG to be 1, the number expressed by the quantity of AF should be that which is said to be $\sqrt{2}$; AB, moreover, will be 2.

Therefore if a *Scale* AB is divided in two equal parts, and four, eight, sixteen, etc., and



[*Fig. 8*]

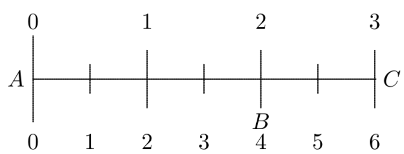
[Fig. 8]

further subdivided as much as one pleases, then certainly any line smaller than the scale can be applied to the scale and thus expressed explicitly by numbers, indeed whether exactly or approximately. *Exactly* when, having begun at the start of the scale A, it falls on some point of division L, as in AL, its number is thence obtained in parts of the scale. For example, with AE being 1, then AL is $\frac{3}{4}$, and then the line AL is commensurable with the scale, that is, there is a common measure for them, or i.e. the line AN ($\frac{1}{4}$), which forms by repetition or i.e. *measures* the scale AB as well as the line AL. But a line applied to a scale can be *approximately* expressed explicitly by numbers when it does not fall on a point of division however much the scale is subdivided and in whatever way the division is established. And a line such as AF is incommensurable with the scale, to the extent that it cannot be expressed explicitly by a rational or 'effable' number, except approximately. Since it is nonetheless necessary that AF, applied to the scale or i.e. translated to AP, at least falls between two division points, so indeed it falls between 11 and 12, supposing the scale AB to be divided into sixteen equal parts, any of which is the eighth part of unity or the foot AE. Hence if AG or AE, the side of a square, is a foot, then the diagonal AF or AP falls between $\frac{12}{8}$ (or $\frac{3}{2}$) and $\frac{11}{8}$ of a foot, and thus if you allot $\frac{3}{2}$ to AF or i.e. $\sqrt{2}$ or y, it is too much, and if $\frac{11}{8}$, too little is allotted (for the square of $\frac{3}{2}$ is $\frac{9}{4}$, that is, more than $\frac{8}{4}$ or 2 or yy, and the square of $\frac{11}{8}$ is $\frac{121}{64}$, which is less than $\frac{128}{64}$ or 2); nonetheless the error is smaller than a smallest part into which unity in the scale is divided here, that is, less than $\frac{1}{8}$. And $\frac{1}{8}$ or AQ will be approximately a common measure for the unity (or indeed the scale) and AF. The more a scale is subdivided, the smaller the error will be, and so it will be as small as anyone desires, or i.e. it can be rendered smaller than any assignable error. Therefore even if AF and AG are incommensurable, they are nevertheless *homogeneous* or comparable, and one can discover a common measure sufficiently precisely that the error or remainder is smaller than a given quantity. And in fact this is the foundation of *approximations*, and *computing by Tables*, and likewise the binary or sexagecimal or even *decimal* Logistic, if indeed a scale is divided into ten parts, and any of them subdivided into another ten, continuing as much as desired. In fact, even if one cannot always divide the scales sufficiently by instruments, nevertheless, by mind or by calculation, one can at least proceed to the highest precision that could be wanted in *practice*. How this is done will appear later.

[Start excised portion.]

A *ratio* is something homogeneous with equality, and so if equality is viewed as the unit, since then the subsequent will be contained in the antecedent only once, the ratio will be the number which arises from the division of the antecedent by the subsequent [per consequens], or i.e. the number by which the antecedent would be expressed if the subsequent were the primary measure by which others should be expressed, or the Unit. Even if possibly another primary measure or unit is now adopted. Thus the ratio of three feet to a foot is triple the ratio which a foot has to itself, or i.e. triple the unit; that is, ternary. Indeed if a foot is 1, three feet will be 3. Unity, moreover, is the ratio of a thing to itself, or to an equal, when the one quantity is contained in the other only once. But the Ratio of one foot to three, that is, to one triple-foot [tripode] (three feet being taken for the unit) is a sub-triple, or i.e. the third part of the ratio which the triple-foot has to itself, or the third part of the unit. Indeed if the triple-foot is 1, the foot would certainly come out as 1/3 of the unit, namely of the triple-foot. The ratio of the foot to the double-foot [dipode] is the sub-double, or half of the ratio which the double-foot has to itself, that is, of the unit. It is therefore 1/2. For instance, if the double-foot is 1, the foot would certainly come out as 1/2 of the unit, namely the double-foot. And the ratio of three feet to the double-foot, or i.e. 3 to 2, is thrice the ratio of one foot to the double-foot, or triple the sub-double, or three of the half, or 3/2. And thus if the double-foot is 1, then three feet would certainly come out as three of the half, or 3/2, of one double-foot.

Proportionals are those of which the ratio is the same.

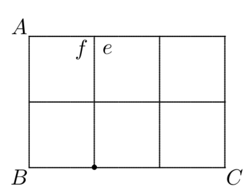


[Fig. 9]

As [the ratio] of the number 3 to 2 is the same as that of the number 6 to 4. For instance, the ratio of the line AC to the line BC is always the same, and therefore the ratio of the numbers by which AC and BC are expressed will also be the same even if a different unit is taken.

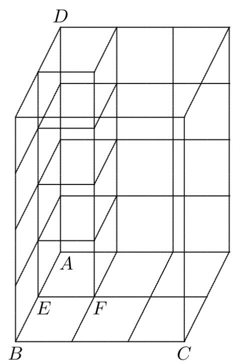
[End excised portion.]

Dimensions are distinct (sometimes heterogeneous) quantities that are understood to be multiplied together, so that the whole of one is applied [applicetur] to each part of the other. For example, by drawing a latitude AB of two inches to a longitude BC of three linear inches (so that the angles A, B, C, e, f are always congruent, or i.e. are the same, what are called right, which is the simplest way of drawing a straight line to a line) the rectangle ABC is formed, which has two dimensions, and six inches, but square.



[Fig. 10]

By drawing a longitude $CB, 2$, latitude $BA, 3$, and altitude $AD, 4$, to each other, a solid rectangle $CBAD$ is formed (Fig. 10), which has three dimensions, or twenty four cubic inches ($2 \,\widehat{\text{in}}\, 3 \,\widehat{\text{in}}\, 4$); indeed on any of the six quadrets or square inches of the base ABC (like the quadret AEF) there stand four cubes, or cubic units, or cubic inches (of course a column or prism $FEAD$ consisting of four cubic inches placed on each other). Nor should one think, as has been believed until now, that dimension is observed in figures alone, and thus that something of higher degree, or i.e. of more dimensions than three dimensions, is imaginary. Even though space *per se* has only three dimensions, a body can nevertheless have many more, for example two bodies, one of gold, another of silver, have, beyond the consideration of the bulk or space that they occupy, yet another consideration of specific gravity, which is observed in any part of the bulk. Thus, setting the specific gravity of a cubic inch of silver at 55, and of gold at 99 ounces (that is nearly the proportion), the weight of the solid $CBAD$, if it is gold, will be $2\frown 3\frown 4\frown 99$ (or $24\frown 99$) or 2376 ounces; but if the solid is silver, it will be $2\frown 3\frown 4\frown 55$ ounces, or $(24\frown 55)$ 1320 ounces.



[Fig. 11]

Therefore the weights have four dimensions, by multiplying the three dimensional bulk or space by the body itself, or the weight. Further, besides the dead weight, there is added a continued heavy impetus from descent over some time; from this arises an impact that has five dimensions: from the three dimensional bulk, the weight of the body, and the elapsed time, multiplied by each other. In this way if one square arm of cloth is worth three imperial coins, two arms will be worth twice three imperials, or six. And this price has two dimensions, for if the same cloth is four arms on the side, its price will be $2 \cdot 3 \cdot 4$ or 24 imperials, and thus of three dimensions by multiplying the longitude, latitude, and price by each other, that is the price, or intrinsic value, to the quantity, or extrinsic value. In this way the price of an agger has four dimensions; one sees, indeed, that if it has a longitude of 100 feet, a latitude of 12, altitude of 20, and hardness or intrinsic value such that a cubic foot is worth ten coins, then its value will be $100\frown 12\frown 20\frown 10$ coins or 240000, whence the drawing of a dimension to a dimension is a real exhibition of mental multiplication.

From these definitions, some Axioms can be derived in sequence.

Those which are determined by the same things (or by coincidents, and in the same manner, of course) are coincident. So two lines are coincident whose two extremes are coincident.

Those which coincide, so much the more are congruent; or i.e. the same thing is congruent to itself.

Those which are determined by congruents (in the same manner, of course) are congruent. So, since a triangle is given when its three sides are given, then if the three sides of a triangle are respectively congruent to corresponding ones, the triangles will be congruent.

Things which are congruent, so much the more are equal.

Equals are expressed by the same number, with the same measure being taken, or i.e. they are of the same quantity; indeed since they can be rendered congruent to each other, they can be made to overlay in the same way on the same primary measure, or unity, repeated in the same manner. The same number results from this.

Equals handled in the same manner according to quantity yield [exhibent] equals.

Similars handled similarly yield [exhibent] similars. Things determined similarly by similars are similar. I understand to be determined, moreover, things that are designated by the same conditions which can happen simultaneously only in one thing. And so, what is thus determined is plainly exhibited.

Things simultaneously similar and equal are congruent. Indeed nothing remains by which they could be distinguished, whether individually or observed simultaneously, unless they are referred to something external, such as place and time and other accidents.

Ratio exists only between homogenea; that is clear from the definition.

[Two] things, one of which is greater than, less than, or equal to the other, are homogeneous. That is manifest for equals, since they can be rendered congruent and thus also similar. The less is also homogeneous to the greater, since it is equal, and thus homogeneous, to its part, and the part is moreover homogeneous to the whole. And for that reason we will not say that a curve, or its part, is less than a surface, nor that an angle of contact is part of a straight line, or less than it. If someone nevertheless considers a part more broadly as everything that has quantity and is-in something having quantity, he can say that a line is less than a surface.

A part is less than the whole. Indeed it is equal to a part of it, namely to itself.

The whole is equal to all the parts integrated [together], indeed they coincide; or certainly if [the parts] are conjoined, since they constitute the whole, they will be rendered coincident [to it], and thus also congruent.

A part of a part is a part of the whole. And so, something less than the lesser is less than the greater. Indeed, it is equal to a part of the lesser, therefore also to a part of the greater, that is, to a part of a part of the greater.

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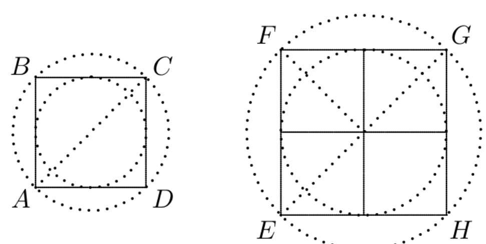
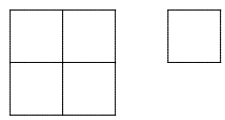
The homologous [components] for one of [two] assumed similar [figures] have the same ratio to each other.



[Fig. 12]

For example, the squares $ABCD$ and $EFGH$ are similar. In the former square, we have side AB. The boundary $ABCDA$; the diagonal AC. An inscribed circle. The circumference of a circumscribed circle. Area of the square. Area of the circle. In the other square we have correspondingly: Side EF. Diagonal EG. Boundary $EFGHE$. The circumscribed circle. Its area. The area of the square itself. We could also inscribe Circles in either case, and implement many other things. Hence I say that the side in the one is in the ratio to its diagonal in which the side of the other also is to its diagonal. For otherwise someone who is in the one square, and afterwards separately in the other, could distinguish them, noting the ratio of side and diagonal in the one to be different from that which is in the other. We have defined similars, however, as those which cannot be distinguished when viewed separately. For the same reason the periphery of one circle is to its diameter as the periphery of the other Circle is to its. Likewise the area of one circle is to the square of its diameter as the area of the other circle also is to the square of its diameter. Hence we could immediately infer the peripheries to be to each other as the diameters, and the circles as the squares of the diameters. And in the same way Spheres [to be] as the Cubes of diameters; and the homologous sides of similar triangles to be proportional if the permutation of ratios had been already demonstrated. And thus, with the aid of this axiom, a great many Geometrical theorems, proven with such great effort by others, are demonstrated with no trouble, and we have a new principle of discovery. While Euclid was forced to demonstrate that circles are as the squares of the diameters, and spheres as the cubes, through reductio ad absurdum, Archimedes, on the other hand, assumed without proof that the centers of gravity of similar figures are similarly placed. All of these arise of their own accord from our definition of similarity, which as far as I know has not occurred to anyone until now. The same principle is valid not only in Geometry, but also in all other [matters] where which quantity and quality are combined.

Heterogeneous things should not, however, be compared with each other, neither is there any ratio except between homogeneous things, otherwise absurdity would arise. For example, if the side AB were to be to the area of its square $ABCD$ as the other side EF is to the area of its square $EFGH$, then (according to things which will be shown in their own place), by permuting, sides AB and EF would be to each other as the squares $ABCD$ and $EFGH$.



[Fig. 13]

Which is absurd; for if, for example, EF is the double of AB, then the square of EF is not the double but the quadruple of the square of AB. And the cube of [a segment] twice as long is not the double but the octuple of the cube. It is the same in circles and spheres as in squares and cubes.

Equimultiples have the same ratio as that of simples, which is clear from what we have said in defining proportionals. For six feet are the same ratio to three which two tripodes are to one. Since the tripode is the same as three feet. Moreover their ratio is also the same if they are expressed in various ways accordingly as one or another unit or primary Measure is assumed.

Equidivisibles have the same ratio as the [corresponding] integers. For the integers are equimultiples of the equidivisibles.

Hence both equimultiples and equidivisibles have the same ratio. Let there be two, A and B; double A will have the same ratio to double B as A to B. Likewise the third part of A will have the same ratio to the third part of B as A to B, and two thirds part of A will have the same ratio to two thirds of B as A to B.

A ratio is composed with a ratio if the antecedent is multiplied by the antecedent and the consequent by the consequent; it is called the composite ratio of simple factors; thus the ratio of the area of one rectangle to another is in the composite ratio of the lengths and widths. Hence the composition of ratios is the multiplication of ratio by ratio.

The *ratio* of A to C is called *composite* of the ratio of A to B and B to C. Hence the ratio of the product of multiplying A by B to the product of multiplying B by C is composed of the ratios of A to B and B to C. For the ratio of the product of A by B to the product of B by C is as A to C (since they are equimultiples, A by B and B by C, therefore they have the same ratio as the simples A and C by the preceding) and the ratio composed of A to B and B to C also is A to C.

Hence ratios composed from the same things are the same. Let ratios A to B and B to C be given, and the ratio L to M and M to N. And let the ratio of L to M be the same as the ratio of A to B, [and] indeed the ratio of M to N the same as the ratio of B to C. Therefore the ratio of A to C will be the same as the ratio of L to N. Since those which are the same or are determined in the same way by the same coincidents coincide. Hence it follows that even if the order of the ratios of the coincidents in one composition were otherwise than in another, the composites would nevertheless coincide.

[End excised portion.]

Two homogenea have an approximate common measure of any desired precision. We showed this above while explaining the scale. If one of two homogeneous things is neither greater nor less than the other, it is equal. In the scale set above (Fig. 8), let AG and AE be compared, and AG applied to the scale, with the point A remaining fixed, the point G will fall between B and A, supposing that AG is less than AB. We now suppose it demonstrable that with the line AG translated to AB, and the point A remaining fixed, the point G falls neither between E and B, nor between A and E, that is, that AG is neither greater nor less than AE; then certainly the point G will fall on the point E itself, and thus AG will be equal to AE. What can be demonstrated of two lines, the same can be done of any homogenea, for they can all be rendered similar, and when they are rendered similar, if they don't differ in magnitude, there is no way in which they can be distinguished, but rather they will be congruent, and so since they can be rendered congruent, they are equal.