A Sample of Figurate Analysis in the Elements of Geometry

G. W. Leibniz, 1685-1687*

Translation and notes by David Jekel†and Matthew McMillan‡

I call figurate analysis that which affords a method of representing figures by letters signifying points, and of discovering and demonstrating their effects and properties, so that not only magnitudes, as in the algebraic calculus, but also situations themselves are directly exhibited by this new type of calculus. We give now a sample of this technique in the *Elements* of Euclid, and in the process we will assume the lemmas, axioms, definitions, and other propositions we need.

Now whatever is explained in the first 10 books of Euclid, this is to be understood *to be in one plane* — so we won't need to be constantly reminded of this.

For Book I of the *Elements*

Proposition 1. Over the given straight line with endpoints AB, to construct the equilateral triangle ABC.

(1) $AC = AB$ by hypothesis. (2) $BC = BA$[1] by hypothesis. And what satisfies these is C. (3) Let $AM = AB$. (4) Then (by Postulate 1[2]) $\overline{M}$ would be *the circumference of a circle with center A and radius*[A] AB (by Definition 1). (5) Let $BR = BA$. (6) Then $\overline{R}$ would be the circumference of a circle with center B and radius BA (as in 4). (7) Now some R is M (by Lemma 1). (8) I.e. $\overline{R}$ and

*Date range inferred from the watermark.

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[1] The manuscript appears to have an A with ink missing at the top of the left stroke, giving the appearance of an H.

[2] Leibniz collected two lists of postulates, definitions, and axioms in the margins of the first two manuscript pages. We have provided the two lists in boxed portions.

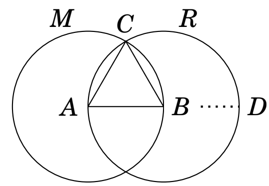

Figure 1

$\overline{M}$ meet each other (from 7 by Definition 2). (9) Given curves $\overline{R}$ and $\overline{M}$ (by 4 and 6) meeting each other (by 7, 8), one has their meeting point[B] (by Postulate 2), namely an R which is an M. (10) But this is C (by 1 and 3 and by 2 and 5). (11) We therefore have C. (12) AB is already given (by hypothesis). (13) Therefore we have ABC. Q.E.F.[C]

Definition 1: Circumference, center, interval. Book 1, Proposition 1, item 4.
Postulate 1: To describe a circle with given center and interval.
Definition 2: The meeting. Item 8.
Postulate 2: Given things that meet, there is a meeting. Item 9.
Definition 3: To be inside a circle. Item 16.
Postulate 3: A line can be extended from one endpoint to any distance whatever. Items 17, 26.
Axiom 1: The whole is greater than the part. Item 20.
Definition 4: To be outside a circle. Item 21.
Axiom 2: A continuum that is inside and outside a figure meets its circumference. Item 24.
Definition 5: A straight line is the shortest path between extremes, or i.e.[D] the distance of two points. Item 32.
Axiom 3: All parts, having no common part, equal the whole. Item 34.
Axiom 4: What moves is not in multiple places simultaneously. Item 38.

Lemma 1. If two circles MA from A and RB[3] from B each have their center in the other's circumference, B in $\overline{M}$ and A in $\overline{R}$, their circumferences meet someplace, at C. [See Figure 1.[4]]

Supposing the same things as before, (14) $BA = BA$ (per se), (15) therefore some R is A (by 5). (16) And so some R is inside the circle $A.\overline{M}$ (by Definition 3, since we say *a point is inside a circle* if its distance to the center is less than the radius). (17) Let AB be extended from B to D (by Postulate 3) (18)

[3] Leibniz wrote RA instead of RB, apparently a mistake.

[4] References to figures (in brackets) have been added to help the reader. Labels and numbers have also been added to the figures (without brackets).

so that $BD = BA$ (by Lemma 2). (19) Therefore, $AD = AB + BD$ (by Lemma 3). (20) Therefore $AD > AB$ (the whole [is greater than] the part, by Axiom 1). (21) Therefore, D is outside $A.\overline{M}$ (by Definition 4, since we say *a point is outside a circle* if its distance from the center is greater than the radius). (22) Already D is R (by 18 and 5). (23) Therefore, some R is outside $A.\overline{M}$. (24) And so (by 16 and 23) some R is in $\overline{M}$ (by Axiom 2, every continuum $\overline{R}$ which is both inside and outside the figure $A.\overline{M}$, is also inside its circumference $\overline{M}$. *Scholium:* Whence,[5] although it would follow from the pure particulars by virtue of form, yet by virtue of matter in continua, 24 follows from 16 and 23.)

(25) Therefore, (from 24 by Definition 2 at 8) $\overline{M}$ and $\overline{R}$ meet each other. Q.E.D.

Lemma 2. Line AB from center B (indeed from any point inside the circle) can be extended so that it also meets the circumference of the circle $\overline{R}$ someplace, at D.

(26) For it can be extended to an arbitrarily large distance (by Postulate 3 at 17). (27) Therefore [it can be extended] to E such that $BE > BA$. (28) Therefore, (by Definition 4 at 21) the line has been extended outside the circle $B.\overline{R}$. (29) The same [line] is in the circle at its center (by Definition 3 at 16). (30) Therefore (by Axiom 2 at 24[6]), it meets its circumference someplace, at D.

Corollary of Lemma 2.[7] A line AD passing through a point B inside a circle meets the circle twice, at A and D. For the line AB extended from B (moving away from A) meets the circumference someplace, at D, by Lemma 2. And DB extended from B (receding from D) will meet the circle someplace, at A.

Lemma 3.[8] If three points $A.B.D.$ are on a line, the distance of some two of them, [say] AD, coincides with the sum $AB+BD$ of the distances of the third[9] B from A, D.

(31) The three points are on a line (by hypothesis). (32) The *line* AD is the shortest path or the distance between the extremes A and D (by Definition 5).

[5] An illegible word follows here and appears to have been crossed out.

[6] Manuscript has "at 23".

[7] In the manuscript, this is written in the margin to the right of Lemma 3.

[8] Leibniz had two versions of this lemma and proof. In the first version, A and B were the endpoints, while in the second version, A and D were the endpoints (however, in the first version he still wrote $AB + BD$ inconsistent with his setup). Then Leibniz crossed out the proof in the first version and the statement in the second version. Thus, we have opted to switch B and D where appropriate to make the statement and proof consistent.

[9] In the Latin, the word "third" grammatically matches "sum" rather than "D"; we believe this may be a typo, but in any case the meaning of the overall statement is clear.

(33) Therefore the point B on the line is less distant from the extreme A than the extremes A and D are from each other. (34) And $AB + BD = AD$ (all the parts having no common part [are equal] to the whole by Axiom 3). (35) But AB is the distance between A and B, and BD the distance between B and D (by Definition 5 at 32). (36) It remains for us to show that, when three points exist on a line, one can get a line which ends in two of them while including the third. (37) Indeed let us suppose that some point runs through the line which they are-in. (38) It will reach them successively (by Axiom 4). (39) Therefore let A be the first, B the second, D the third. (40) Therefore the portion of the line it traverses between A and D will terminate in A and D but include B.

Addition 1. If two circles $A\overline{M}$, $B\overline{R}$ of equal radii have radius greater than half the distance of their centers $A.B$, they will meet each other in C outside the line through the centers.

(41) The line AB intersects $\overline{R}$ twice, at E and D (by the corollary of Lemma 2). (42) And similarly $\overline{M}$ at F. (43) Let E be [on the line] from B toward A (44) and let D be [on the line] from B moving away from A (45) and let F be [on the line] from A toward B, (46) AE will be less than AF (see 58 shortly).[10] (47) Since (in view of 43) E falls between B and A, or B between E and A, (48) AE will be (by Lemma 3) the difference between AB and BE, (49) or between AB and AF (50) since $AF = BE$ (by the hypothesis). (51) Now if $AF > AB$ (52) we will have $AF = AB + AE$ (by 48, 49). (53) Therefore, $AF > AE$. (54) But if $AB > AF$, we will have $AB - AF = AE$ (by 48, 49). (55) Now $AF > \frac{1}{2}AB$ (by the hypothesis). (56) Therefore $AE < \frac{1}{2}AB$. (57) Therefore $AF > AE$. (58) Either way, therefore, $AE < AF$ as was asserted in item 46. (59) In turn, AD is greater than AF. (60) For $AD = AB + BD$ (by Lemma 3) (61) $= AB + AF$ (because $BD = AF$[11] by hypothesis). (62) Therefore some R is inside $\overline{M}$, namely E (because $AE < AF$ or AM by 58). (63) Some R is outside $\overline{M}$, namely D (since by 59, $AD > AM$ or AF[12]). (64) Therefore (by Axiom 2) some R is in $\overline{M}$, say C.

Addition 2. Above a given base AB, construct an isosceles triangle whose legs AC or BC are of a given magnitude G, which should be greater than half the base AB. [See Figure 2.]

(65) With centers A and B, (66) interval $= G$ (by Proposition 2 demonstrated independently of this), (67) let circles be described (Postulate 1) which intersect someplace, at C. By Addition 1 we will have $AC = G$ and $BC = G$. Q.E.F.

[10] The parenthetical was added in the margin.

[11] Manuscript has "$BD = BF$" but the mathematical argument suggests it should be $BD = AF$.

[12] Manuscript has "AM" but it seems that "AF" was intended.

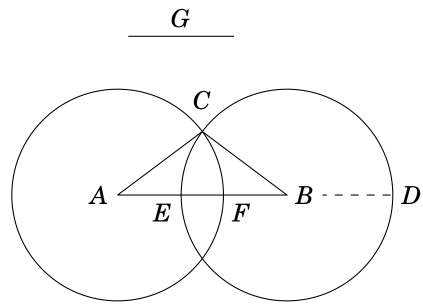

Figure 2

Proposition 2.[E] At a given point A place a line AG equal to a given line BC. [See Figure 3.]

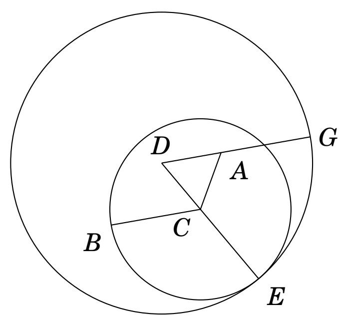

Figure 3

Solution. (1) Connect AC. (2) Let equilateral triangle ACD be constructed upon it (by [Proposition] 1 of the first [Book][13]). (3) Let a circular circumference BE be constructed with center C, radius BC (by Postulate 1) (4) which the line DC extended from C (by Postulate 3) (5) will meet someplace, at E (by Proposition 1, Lemma 2). (6) Let a circle be described with center D, radius DE (Postulate 1) (7) which the line DA extended from A will meet someplace, at G (by what was said in Lemma 2). (8) We will have $AG = BC$. (9) Indeed $DC + CE = DE$ (by 5 and Lemma 3 for Proposition 1). (10) $= DG$ (by 6 and 7)

[13] Leibniz's reference to Euclid is the rather elliptical *1. prim.*

(11) $= DA + AG$ (by 7 and Lemma 3 for Proposition 1). (12) $= DC + AG$ (by 2). (13) Therefore (by 9 and 12) $AG = CE$ (14) $= BC$ (by 3, 4).

Scholium to Proposition 2.[14] The analysis by which this construction can be discovered is of this sort: A line is to be placed at point A equal to the line placed at point C. By a line placed at point C one can understand not only BC but also any other such as CE, equal to CB, or drawn from C to the circumference of a circle CBE. Since, therefore, points A and C ought to be treated in the same way, the line AC should also be treated in such a way that with respect to C it is treated just as it has been treated with respect to A. Therefore, let some point D be sought relating in the same way to A and C, which arises by constructing an equilateral or isosceles triangle ADC. The line drawn from D through C will meet the circle in E; in the line from D extended through A let DG be taken equal to DE, and we will have $AG = CE$, because G is found in the same way with respect to A as E with respect to C.

Proposition 3. Given two lines, A and a larger BC, from [BC] subtract BE equal to A. [See Figure 4.]

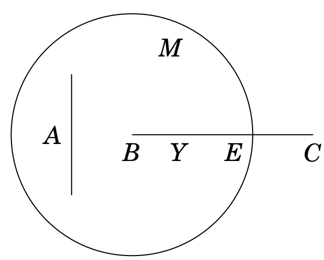

Figure 4

(1) Make $BD = A$ (by [Proposition] 2[15] of the first [Book]) (2) and make $\overline{M}$ such that $BM = BD$ (by Postulate 1). (3) Now BC is inside $\overline{M}$ at B (as at item 16 of Proposition 1) (4) and outside $\overline{M}$ at C ((5) because $BC > A$ by hypothesis, (6) hence $> BM$ by 1, 2). (6) Therefore (by Axiom 2) BC meets $\overline{M}$ someplace, at E. (7) Therefore BE is a part of BC. (8) And $BE = BD = A$.

Also thus:

(3) Let $\overline{Y} \infty BC$. (4) We will have $BY + YC = BC$ (by Lemma 3 for Proposition 1). (5) $BC > A$ (by hypothesis). (6) Therefore some $BY = A$ (indeed by

[14] Only “Schol.” is underlined in the manuscript, but like the Corollary to Lemma 2, we format the entire title.

[15] Leibniz’s reference *3. prim.* seems to indicate the wrong proposition.

Definition 6: If A is equal to a part of B, then A is said to be *lesser*, B *greater*. Book 1, Proposition 3, item 6.
Definition 7: Rectilinear angles are said to be *equal* if they are congruent. Proposition 4, item 3.
Axiom 5: Given the points at which the angles of a figure stand, the figure is given. [Proposition 4,] item 9. This can be considered a postulate.
Axiom 6: Those which are given (determinately) in the same way from congruent givens are congruent. [Proposition 4,] item 11.

Definition 6. Greater and lesser, A is *less* when some part BY of the other, BC, which is called *greater*, is equal to it). (7) Therefore some Y is M. (9) Let it be E. Therefore some E is given (by Postulate 2). (10) Therefore also $BE = A$ (by 6), (11) a part of BC (by 4). Q.E.F.

Proposition 4. If two Triangles BAC, EDF have two sides of the one, BA, AC equal to two corresponding sides of the other, ED, DF respectively, and the angle A of the one equal to the angle D of the other, contained by the equal straight lines, then the triangle will be congruent to the triangle. [See Figure 5.]

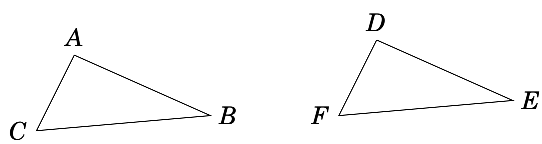

Figure 5

(1) Let us suppose that the angles are given in position, LAM and NDP, (2) equal by the hypothesis, (3) hence congruent. (For *equal rectilinear angles* are defined in *Definition 7* as those which are congruent.) (4) And [suppose] G has the magnitude of AC and DF (5) and H the magnitude of AB and DE. (6) Finally it is given on which sides of the angles these lines are to be taken, namely AB on AM, AC on AL, DE on DP, DF on DN. (7) Therefore points B, C, likewise E, F are given (by 3 of the first [Book]). (8) Therefore A, B, C [are given]; likewise D, E, F (by 1 and 7). (9) Therefore, triangles ABC and DEF [are given] (for, given the points at which the angles of the figure stand, the figure is given by *Axiom 5*). (10) And indeed both are exhibited determinately in the same way from congruent givens (by the whole procedure). (11)

Therefore they are congruent (by *Axiom 6*). Q.E.D.

Scholium. If one applies superposition, it comes to the same thing; indeed, if the congruent givens become actually congruous, or coincident by superposition, then those which are given determinately from them will also coincide; otherwise not one but several things satisfying these givens could obtain, contrary to hypothesis. From this one understands the reason for the sixth axiom.

Porism.[F] A triangle is given in magnitude and shape [species] if an angle and the sides enclosing it are given in magnitude (by 1 through 9). Indeed, for it to be given in position, one only needs an angle to be given in position, as well as which legs of the angle are assigned to which magnitudes of sides, which change nothing in the magnitude and shape, since either leg relates in the same way at the angle.

Scholium. I call *porism* that which is inferred from a demonstration, and *corollary* that inferred from a proposition.

Proposition 5. A triangle ABC that has two sides AB and AC equal also has their two angles B and C on the remaining side BC equal. [See Figure 6.]

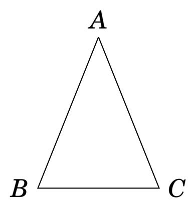

Figure 6

(1) Suppose BC is given in position. (2) And let G be given, equal to BA, CA (3) and the *side* on which[16] A should fall (this is by *Definition 8*:[17] the plane being cut into two parts by the line BC extended indefinitely, let it be given in which part A should be). (4) A is given if it is possible, (5) for with centers B and C and interval equal to G (by 2 of Book 1) (6) circles are described (Postulate 1). (7) Now A is in the circumference of both (by 2), if it is possible, of course. (8) Therefore the circles meet each other at A if A is possible. (9) Therefore (by Postulate 2) A is given. (10) Therefore (by Axiom 5) ABC is given. (10) Therefore also the angles ABC, ACB (since by Axiom 7 when something is given, its requisites are given) (11) and indeed [given] in

[16] Literally "partes ad quas". See also item 1 of Prop. 6 below.

[17] This definition is apparently missing from the manuscript.

the same manner (by the procedure explained). (12) Therefore (by Axiom 6) the angles are congruent. (13) And hence equal. (For congruents are equals by Axiom 8.)

Scholium. The same thing could have been shown by superposition, if some triangle DEF had been taken congruent to this one, and now ABC would be placed onto DEF, now ACB onto DEF; thus now angle ABC, now angle ACB, would agree with the same DEF; therefore, they would be congruent to each other.

Proposition 6. A triangle ABC that has two angles B and C equal will also have the two sides AB and AC belonging to the remaining angle A equal.

This is demonstrated in the same way, (1) since, given side BC and angles B, C and the side on which A should be, A is given. (2) Indeed, line BC and the angle to it of another line BA being given in position, the line BA itself is given indefinitely (for the angle being given in position, by Axiom 7 the sides are given), in the same way CA indefinitely; (3) these will intersect each other in A if A is possible. (4) Therefore A is given, and thus both [sides are given] in the same way, therefore they will be congruent. Q.E.D.

Scholium. The same thing could have been demonstrated from the preceding. As also by superposition in the manner of the preceding.

Proposition 7. If triangles ABC, DEF have sides equal to the corresponding sides of the other, the triangles will be congruent. [See Figure 7.]

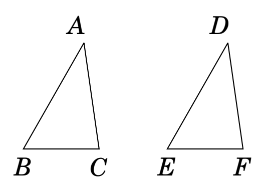

Figure 7

This is demonstrated by the same method, since, one side of one triangle being given in position, and the one equal to the other, and the remaining ones being given in magnitude, by describing circles from the extremes of the sides given in position and with the magnitudes as intervals, each of the triangles will be given, in the same way; therefore by Axiom 6 they will be congruent.

Proposition 8. To bisect a given angle BAC. [See Figure 8.[18]]

[18]Leibniz originally had the lines from A to F, B to F, and C to F dashed, but then he wrote over them with a solid line.

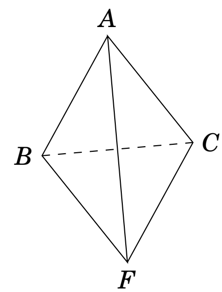

Figure 8

Let FA be the bisecting line, and $FAB = FAC$. It relates in the same way to BA and CA. We have one point of the line FA, namely A, let us further seek some F to which this line relates as to A. This will happen if (supposing BA, CA are equal so that each side is treated in the same way with respect to FA) we translate BAC into BFC by describing circles with centers B and C and radii equal to BA or CB, which intersect each other someplace, at F (since triangle BFC is congruent to BAC by 7 of Book 1 on account of the same base BC and equal sides, certainly it is possible). Therefore line FA, relating in the same way to AB and AC, will certainly bisect the angle.

The same will obtain if we construct any isosceles triangle BFC whatsoever over the base BC of the isosceles triangle ABC, by Addition 2 to Proposition 1. For the locus of all points relating in the same way to the two sides of the same angle or to the two extremes of the same line is a line passing through the two apices of the two isosceles triangles relating in the same way to the proposed angle or line.

Proposition 9.[19] To bisect a given line BC. [See Figure 9.]

Two points should be sought relating in the same way to B and C. This will happen if two isosceles triangles BAC, BDC of any sort are constructed over the base BC; the line drawn through the angles opposite the base, or through their apices, will bisect the base.

Scholium. Euclid uses an equilateral triangle for Propositions 8 and 9, but it is better to use a more general construction.

[19]Manuscript has “Prop. 8” which repeats the number 8 for no clear reason; likely a mistake.

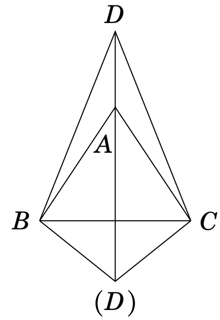

Figure 9

Notes

[A]The Latin *intervallo* sometimes indicates the radius of a circle. It may suggest here that the data of the segment is given before the circle is determined.

[B]The Latin *occursus* indicates the intersection point, but in Item (9) Leibniz crossed out *intersectio* in favor of *occursus*, i.e. *meeting*, suggesting some difference of connotation.

[C]*Quod Erat Faciendum*, i.e. *that which was to be made*. Originally "Qu. Er. Fac.". We standardize Leibniz's various forms of this expression to "Q.E.F." throughout this essay. Similarly we standardize "Quod erat demonstrandum" to "Q.E.D." throughout.

[D]In some places Leibniz defines the distance as the shortest path itself, not the number or physical quantity.

[E]This is known as the "Compass Equivalence Theorem" and establishes the possibility of transferring a segment by a rigid motion.

Straightedge-and-compass constructions are traditionally constrained to those for which the compass cannot be locked and both feet lifted from the tablet at the same time.

Here Leibniz attempts to derive the theorem from his symmetry considerations.

[F]Here the word is "porisma" which is the Greek for "corollary." Literally, "bonus, windfall".