De Determinatis et Congruis

G.W. Leibniz, 1680-82
Tr. David Jekel and Matthew McMillan

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[Manuscript]
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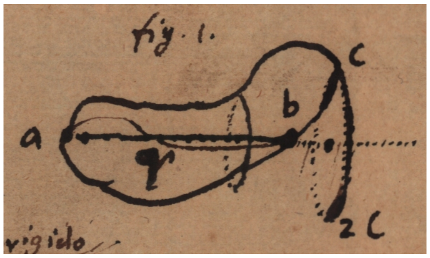
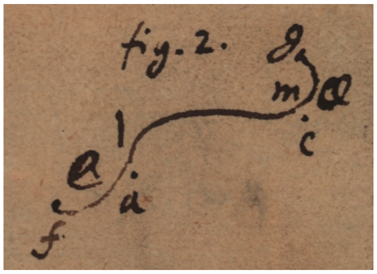
I call a *rigid extensum* that in which, with any kind of motion, no change occurs with respect to extension, that is the extension not only of itself, but also of things that are in it, is not changed.



[Fig. 1]

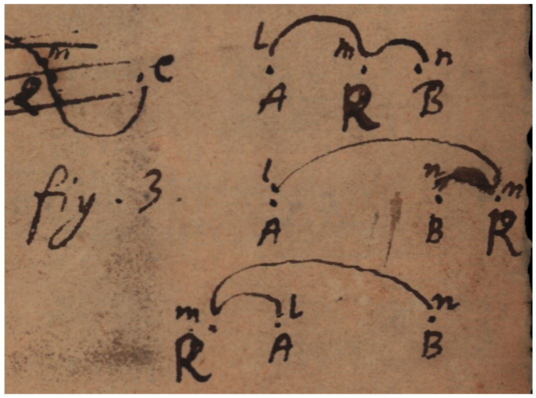
If some rigid Extensum abc is moved with two of its points a and b staying at rest, then the locus of all its points at rest, say r, will be the line ab, while the successive locus $c2c$ of some point c that is moved will be an *arc of a circle*. Therefore, we have the generation of a line and a circle, the line by rest, the circle by motion, without any line or circle pre-existing.

From this generating process [generatio] of the line it follows that any point, say r, is on the same line with the two given points $a.b$, if when connected to them by some rigid extensum arb, it cannot move when they are at rest. And conversely, if some point is on the same line with two given points to which it is connected by a rigid extensum, it cannot move when they stay at rest. Now the points, say $a.r.b$, are connected sufficiently by a rigid extensum when they are all found in the same rigid extensum, say abc.



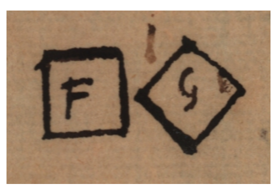
[Fig. 2]

If points such as a and c always preserve the same situs between themselves, then the same points $l.m$ of the same rigid extensum $f\,l\,m\,g$ can always be attached to them; and conversely if the same points of the same rigid extensum can always be attached to them, they always preserve the same situs among themselves.



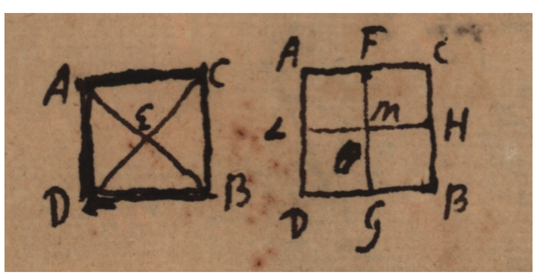
[Fig. 3]

Therefore a line can be defined as the locus of all points, each one of which, say R, has the situs to two points A and B such that as long as it preserves this situs to them (which happens as long as the three points $A.R.B.$ can be attached to the same points $l.m.n$ of the rigid [extensum] lmn), R cannot move with A and B unmoved (that is, if l and n remain unmoved, then no matter how the extension lmn moves, it is also necessary that m is unmoved). And so the locus of this R is determined when its situs to the points A and B is determined. Hence it is immediately clear that the points A and B certainly fall on this same line, or the locus of all points R. For certainly R cannot move when A and B remain unmoved. It is also clear that in an extensum that is unchangeable as far as extension (such is space itself, as well as any rigid body), with two points being determined, the line that passes through them is determined.



[Fig. 4]

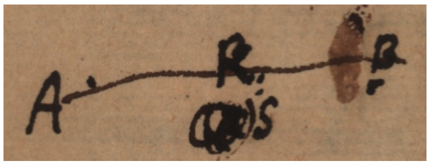
In order to express by calculus this property of the line, I will take ∞ as the symbol for *congruence*, for example $F \infty G$, that is, F is congruent to G.



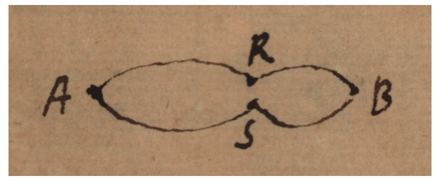
[Fig. 5]

Now ɣ will be the sign of *coincidence*. For instance if in an equilateral square $ACBD$, the lines AB, CD are drawn connecting opposite angles and intersecting each other at the point E, and again other lines FG, HL are drawn connecting the midpoints of the opposite sides and intersecting each other at the point M, then in fact the points E and M coincide, and one can write $E\,\text{ɣ}\,M$.

The *situs* of certain *points between themselves* I write with a simple denomination of the points and dots between, as $A.R$ or $R.A$. This signifies the situs determined between the points A and $R.$, or any rigid thing connecting them. Similarly $A.B.R$ signifies the situs of the three points among themselves. And so on.



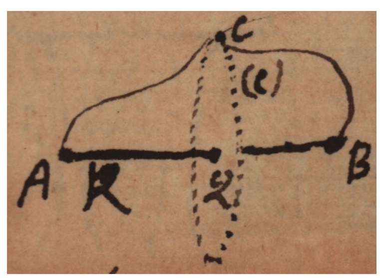
[Fig. 6]



[Fig. 7]

And thus, when this situs of the point R to two points $A.B$, which determines the locus of the point R, is the situs of a point R on a straight Line with points A and B; and its situs to the points A and B is written $A.R.B.$, determined if you like by some rigid thing ARB connecting the three points; and there is moreover some point S having the same situs to the points A and B that the point R has, so that the rigid thing attached to the points $A.R.B$ could be attached in the same way to the points $A.S.B$; [then] it follows that the points S and R coincide if in fact R is on the same line with A and B.

And so, to enfold the matter concisely and with calculus, if supposing $A.R.B \infty A.S.B$ it follows that $R\,\text{ɣ}\,S$, the locus of all R will be a *Line*. And again, if in addition we want another sign for determination as an abbreviation, [then] by writing $\overline{dt\,R}$ we can signify that the locus of the point R is determined or certain, and similarly by writing $\overline{dt\,A.R}$ that the situs of the points A and R is determined or certain. Thus $\overline{dt\,A.R.B}$ will signify that the situs of these three points is certain. And so the *Line* passing through A and B is the locus of any points R whatever, if [such that] from $\overline{dt\,A.R.B}$ it follows that $\overline{dt\,R}$.

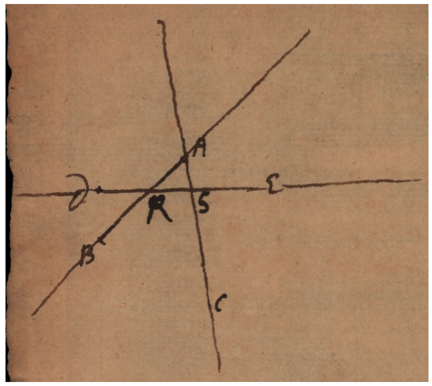


[Fig. 8]

Now all points outside this line can be moved while preserving their situs to the two points A and B. Thus, if one places a point C connected rigidly to A and B, but it is not on the line passing through A and B, it will be able to be moved with the points A and B being unmoved, and by its motion will describe a circle $C(C)$ in the way we defined above; therefore since during the motion it always preserves the same situs to the points A and B, we will have $A.C.B \infty A.(C).B$ and the locus of all C or (C) will be circular.

Every point, such as $C.$, that preserves the same situs to some points such as A and $B.$, will preserve the same situs to that point $R.$ which has a determined place when the situs to A and B is determined (and hence the point C being moved, with the situs to the points A and $B.$ preserved, will preserve the same situs also to any point $R.$ of the line passing through A and B, or i.e. to this line itself); or in general, whatever preserves its situs to the determiners, preserves it also to the determined. This proposition deserves to be demonstrated.

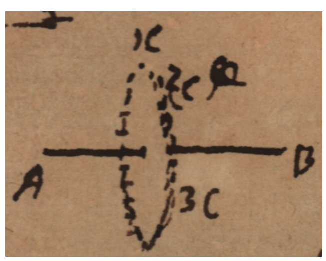
It can also be shown that when one point B is taken on the line AB, another point R can always be found such that $C.B \infty C.R.$,[1] except in a single case, when B and R, speeding toward each other uniformly, meet each other, at Q.



[Fig. 9]

A *plane* is determined when three points are assumed, or i.e. all the points whose place is determined when the situs to three points is given fall on the plane passing through those three points; this definition coincides with the generation of a plane by a line moved over two unmoved lines. Indeed, let there be two lines, one passing through points A and B, the other passing through points A and C. And let there be another movable line passing through points $D.E$. This one, moved over those two lines, and touching on AB at R, AC at $S.$, will give the straight line passing through $R.S.$, or determined from the points R and S. And since all the points such as R and S are determined from the points $A.B.C.$, and the lines connecting them are determined from them, and all points of these lines (from the proposed generation of the plane by the motion of lines) are all the points of the plane, it is clear that all the points of the plane are determined from these three points $A.B.C$.

And infinite space itself is determined by taking four points; or i.e. the locus of all points each of which is determined given its situs to four given points is infinite space, or the locus of all points in general; or what is the same thing, the locus of every point is determined if its situs to four given points is given; but this must be demonstrated.



[Fig. 10]

As it was said above that the point C during its motion relates in the same way to the line AB, therefore the line AB will relate in the same way to the three points $1C.2C.3C$. Therefore it will also relate in the same way to the plane determined by these three points. And in the same way, when three points determining a plane have the same situs to some two points A and B, the determined plane itself will also have the same situs to them.

[1] The manuscript has $C.B. \infty C.B.$.

It should be demonstrated that the same propositions are reciprocal, or that all points having simultaneously the same situs to two distinct points fall on a plane, and [those] having [the same situs] to three fall on a line. With these things obtained, one may join those demonstrated elsewhere about congruents, and those demonstrated here about things determined, and one will obtain that two points, having the same situs to four given points which do not have the same situs to three other points, will agree.