90 **(40950)**. DE DETERMINATIS ET CONGRUIS
[1680 – 1682 (?)]

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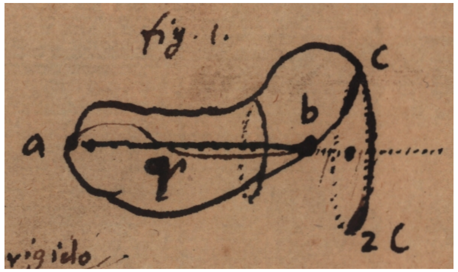
E x t e n s u m r i g i d u m voco, in quo utcunque moto nulla quoad extensionem fit mutatio, id est non tantum ipsius, sed nec eorum quae in ipso sunt extensio non mutatur.



Si Extensum aliquod rigidum $a\,b\,c$ duobus ejus punctis a et b quiescentibus moveatur, tunc omnium ejus punctorum quiescentium, ut r locus erit recta $a\,b$, puncti autem alicujus moti c locus successivus $c\,2\,c$ erit a r c u s c i r c u l i. Habemus ergo generationem rectae et circuli, nulla recta nulloque circulo praeexistente[,] rectae quidem per quietem, circuli per motum.

Ex hac generatione rectae sequitur punctum omne, ut r in eadem cum duobus datis punctis $a.\ b$ recta esse, si cum ipsis rigido aliquo extenso ut $a\,r\,b$ connexum, ipsis quiescentibus moveri non potest. Et vicissim si punctum aliquod in eadem cum duobus datis punctis recta sit, cum quibus rigido extenso connexum est, quiescentibus illis moveri non posse. Satis autem extenso rigido connexa sunt puncta, ut $a.r.b$ quando omnia in eodem extenso rigido, ut $a\,b\,c$ reperiuntur.

De Determinatis et Congruis

G.W. Leibniz, 1680-82
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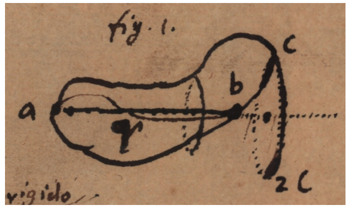
DRAFT, 22 March 2025

[Manuscript]
[Typescript]

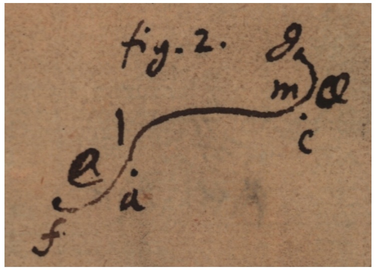
I call a *rigid extensum* that in which, with any kind of motion, no change occurs with respect to extension, that is the extension not only of itself, but also of things that are in it, is not changed.



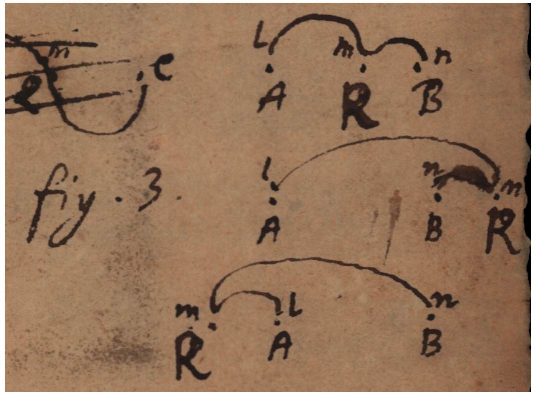
[Fig. 1]

If some rigid Extensum abc is moved with two of its points a and b staying at rest, then the locus of all its points at rest, say r, will be the line ab, while the successive locus $c2c$ of some point c that is moved will be an *arc of a circle.* Therefore, we have the generation of a line and a circle, the line by rest, the circle by motion, without any line or circle pre-existing.

From this generating process [generatio] of the line it follows that any point, say r, is on the same line with the two given points $a.b$, if when connected to them by some rigid extensum arb, it cannot move when they are at rest. And conversely, if some point is

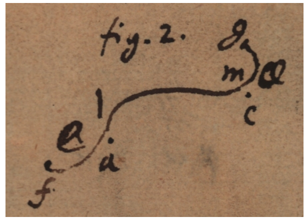


Si puncta ut a et c eudem semper servant situm inter se, tunc eadem semper ejusdem extensi rigidi $f\,l\,m\,g$ puncta $l.m$ eis applicari possunt; Et vicissim si eadem semper extensi rigidi puncta eis applicari possunt, eundem semper servant situm inter se.



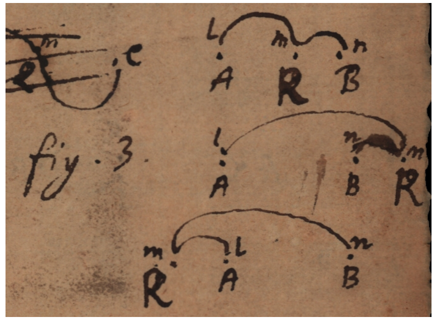
Recta ergo definiri potest locus omnium punctorum, quorum unumquodque ut R eum habet situm, ad duo puncta A et B, ut quamdiu hunc ad ea situm servat (quod fit quamdiu tria puncta $A.R.B.$ iisdem rigidi $l\,m\,n$ punctis $l.\,m.\,n$ applicari possunt) ipsis A et B immotis, R moveri non possit, (id est si l et n manentibus immotis, tunc utcunque moveatur extensum $l\,m\,n$, necesse est et m esse immotum). Itaque locus ipsius R, est determinatus, determinato ejus situ ad puncta A et B. Hinc statim patet ipsa puncta A et B, utique cadere in hanc ipsam rectam seu locum omnium punctorum R. Utique enim ipsis A et B manentibus immotis ipsum R moveri non potest. Patet etiam in extenso quoad extensionem immutabili, (quale est ipsum spatium, item corpus

on the same line with two given points to which it is connected by a rigid extensum, it cannot move when they stay at rest. Now the points, say $a.r.b$, are connected sufficiently by a rigid extensum when they are all found in the same rigid extensum, say abc.



[Fig. 2]

If points such as a and c always preserve the same situs between themselves, then the same points $l.m$ of the same rigid extensum $f\,l\,m\,g$ can always be attached to them; and conversely if the same points of the same rigid extensum can always be attached to them, they always preserve the same situs among themselves.

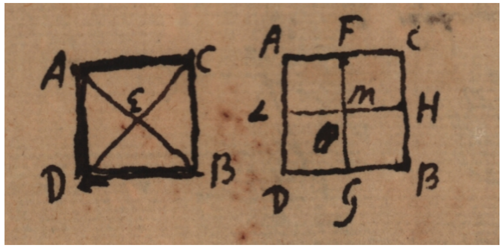


[Fig. 3]

Therefore a line can be defined as the locus of all points, each one of which, say R, has the situs to two points A and B such that as long as it preserves this situs to them (which happens as long as the three points $A.R.B.$ can be attached to the same points $l.m.n$ of the rigid [extensum] lmn), R cannot move with A and B unmoved (that is, if l and n remain unmoved, then no matter how the extension lmn moves, it is also necessary aliquod rigidum) duobus punctis determinatis, determinatum esse rectam quae per ipsa transit.

[*Fig. 4*]

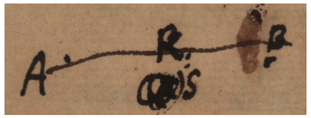
Hanc rectae proprietatem ut calculo exprimam, erit mihi ∞ signum c o n g r u e n t i a e , exempli causa $F \infty G$ id est F congruum esse ipsi G.



[*Fig. 5*]

At ♉ signum erit c o i n c i d e n t i a e. Ut si in quadrato aequilatero $ACBD$, ducantur rectae AB, CD, angulos oppositos connectentes, et se secantes in puncto E, et rursus ducantur aliae rectae FG, HL, laterum oppositorum puncta media jungentes, et se secantes in puncto M, tunc revera puncta E et M coincidunt, poteritque scribi E ♉ M.

S i t u m p u n c t o r u m quorundam i n t e r s e scribam simplici punctorum denominatione interpuncta, ut $A.R$ vel $R.A$. Significat situm inter puncta A et R. determinatum, vel rigidum quodcunque ipsa connectens. Similiter $A.B.R$ significat trium punctorum situm inter se. Et ita porro.



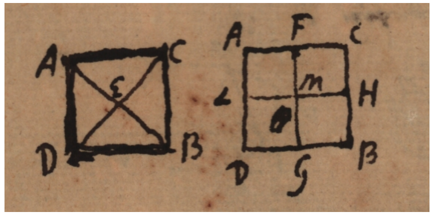
[*Fig. 6*]

[*Fig. 7*]

that m is unmoved). And so the locus of this R is determined when its situs to the points A and B is determined. Hence it is immediately clear that the points A and B certainly fall on this same line, or the locus of all points R. For certainly R cannot move when A and B remain unmoved. It is also clear that in an extensum that is unchangeable as far as extension (such is space itself, as well as any rigid body), with two points being determined, the line that passes through them is determined.

[Fig. 4]

In order to express by calculus this property of the line, I will take ∞ as the symbol for *congruence*, for example $F\infty G$, that is, F is congruent to G.



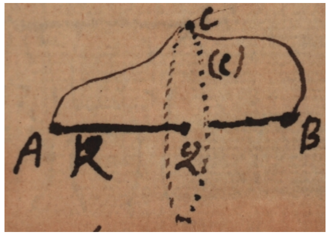
[Fig. 5]

Now ♉ will be the sign of *coincidence*. For instance if in an equilateral square $ACBD$, the lines AB, CD are drawn connecting opposite angles and intersecting each other at the point E, and again other lines FG, HL are drawn connecting the midpoints of the opposite sides and intersecting each other at the point M, then in fact the points E and M coincide, and one can write E♉M.

The *situs* of certain *points between themselves* I write with a simple denomination of the points and dots between, as $A.R$ or $R.A$. This signifies the situs determined between the points A and R., or any rigid thing connecting them. Similarly $A.B.R$ signifies the situs of the three points among themselves. And so on.

Cum itaque situs ille puncti R ad duo puncta $A.B$, qui locum puncti R determinat, sit situs puncti R in linea Recta cum punctis A et B; situsque ejus ad puncta A et B scribatur $A.R.B$. determinatus si placet rigido aliquo ARB tria puncta connectente; sitque praeterea punctum aliquod S, eundem habens situm ad puncta A et B quem habet punctum R, ita ut rigidum punctis $A.R.B$ applicatum eodem modo applicari possit ad $A.S.B$ puncta; sequetur puncta S et R coincidere, siquidem R cum ipsis A et B est in eadem recta.

Itaque ut rem paucis et calculo complectar, si posito esse $A.R.B \infty A.S.B$ sequitur esse R ♉ S erit locus omnium R, R e c t a. Et vicissim sin compendii causa aliud praeterea adhibere velimus signum determinationis, poterimus scribendo $\overline{dt\ R}$ significare locum ipsius puncti R esse determinatum seu certum, et similiter scribendo $\overline{dt\ A.R}$ situm punctorum A et R esse determinatum seu certum. Ita $\overline{dt\ A.R.B}$ significabit situm horum trium punctorum esse certum. Itaque R e c t a transiens per A et B est locus punctorum R quorumcunque, si ex $\overline{dt\ A.R.B}$ sequitur $\overline{dt\ R}$.

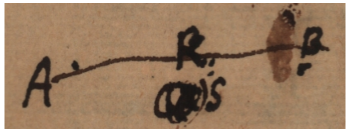


[*Fig. 8*]

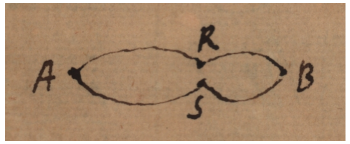
Omnia autem puncta extra hanc rectam moveri possunt servato suo situ ad duo puncta A et B. Ita licet punctum C ponatur ipsis A et B rigide connexum, si tamen non sit in recta per A et B transeunte, moveri poterit punctis A et B immotis, et motu suo describet circularem $C(C)$ quemadmodum supra definivimus; itaque cum durante motu eundem semper situm servet ad puncta A et B, habebimus $A.C.B \infty A.(C).B$ eritque locus omnium C vel (C) circularis.

Omne punctum, ut C. quod eundem situm servat ad aliqua puncta, ut A et B. servabit etiam eundem situm ad id punctum R. quod situ ad A et B determinato locum

8 ∞: Leibniz wechselt zur Bezeichnung der Kongruenz vom Symbol ∞ zum Symbol ∞.



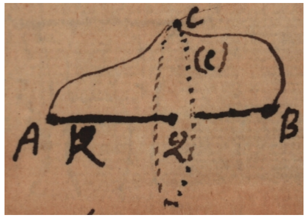
[Fig. 6]



[Fig. 7]

And thus, when this situs of the point R to two points $A.B$, which determines the locus of the point R, is the situs of a point R on a straight Line with points A and B; and its situs to the points A and B is written $A.R.B.$, determined if you like by some rigid thing ARB connecting the three points; and there is moreover some point S having the same situs to the points A and B that the point R has, so that the rigid thing attached to the points $A.R.B$ could be attached in the same way to the points $A.S.B$; [then] it follows that the points S and R coincide if in fact R is on the same line with A and B.

And so, to enfold the matter concisely and with calculus, if supposing $A.R.B \infty A.S.B$ it follows that R♉S, the locus of all R will be a *Line*. And again, if in addition we want another sign for determination as an abbreviation, [then] by writing $\overline{dt\,R}$ we can signify that the locus of the point R is determined or certain, and similarly by writing $\overline{dt\,A.R}$ that the situs of the points A and R is determined or certain. Thus $\overline{dt\,A.R.B}$ will signify that the situs of these three points is certain. And so the *Line* passing through A and B is the locus of any points R whatever, if [such that] from $\overline{dt\,A.R.B}$ it follows that $\overline{dt\,R}$.

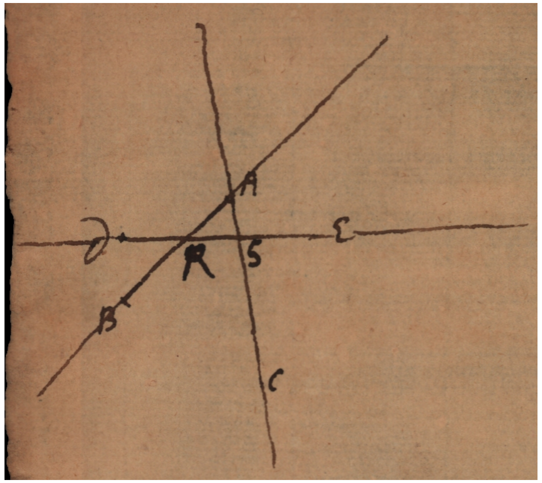


[Fig. 8]

Now all points outside this line can be moved while preserving their situs to the two points A and B. Thus, if one places a point C connected rigidly to A and B, but it is not on the line passing through A and B, it will be able to be moved with the points A and B being unmoved, and by its motion will describe a circle $C(C)$ in the way we defined habet determinatum; (et proinde punctum C motum servato situ ad puncta A et B. eundem servabit situm etiam ad quodlibet punctum R. rectae per A et B transeuntis, seu ad ipsam hanc rectam) seu generaliter quod situm eundem servat ad determinantia, servat et ad determinatum. Haec propositio meretur demonstrari.

Poterit etiam ostendi, semper puncto uno sumto B in recta AB inveniri posse aliud R, ita ut sit $C.B \infty C.R.$ excepto unico casu, ubi B et R ad se properantia uniformiter sibi occurrunt, scilicet in Q.

P l a n u m determinatum est, assumtis tribus punctis, seu omnia puncta quorum situ ad tria puncta dato determinatus est locus, cadunt in planum per illa tria puncta transiens; coincidit haec definitio cum generatione plani per rectam super duabus rectis immotis motam:



[*Fig. 9*]

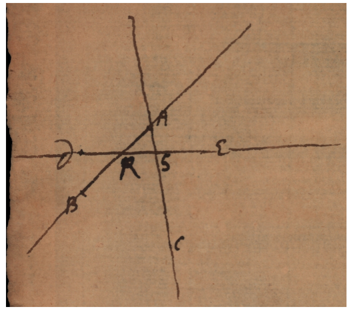
Sint enim duae rectae, una transiens per puncta A et B, altera transiens per puncta A et C. Et sit alia recta mobilis transiens per puncta $D.E$. Haec mota super his duabus rectis, et AB attingens in R, AC in S. dabit lineam rectam per $R.S.$ transeuntem seu ex punctis R et S determinatam.

Cumque omnia puncta ut R et S sint ex punctis $A.B.C.$ determinata, et rectae ipsas jungentes determinatae ex ipsis, omniaque puncta harum rectarum (ex posita generatione

above; therefore since during the motion it always preserves the same situs to the points A and B, we will have $A.C.B \infty A.(C).B$ and the locus of all C or (C) will be circular.

Every point, such as $C.$, that preserves the same situs to some points such as A and $B.$, will preserve the same situs to that point $R.$ which has a determined place when the situs to A and B is determined (and hence the point C being moved, with the situs to the points A and $B.$ preserved, will preserve the same situs also to any point $R.$ of the line passing through A and B, or i.e. to this line itself); or in general, whatever preserves its situs to the determiners, preserves it also to the determined. This proposition deserves to be demonstrated.

It can also be shown that when one point B is taken on the line AB, another point R can always be found such that $C.B \infty C.R.$,[1] except in a single case, when B and R, speeding toward each other uniformly, meet each other, at Q.



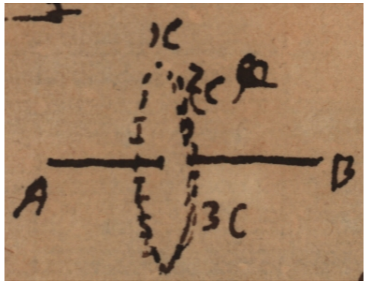
[Fig. 9]

A *plane* is determined when three points are assumed, or i.e. all the points whose place is determined when the situs to three points is given fall on the plane passing through those three points; this definition coincides with the generation of a plane by a line moved over two unmoved lines. Indeed, let there be two lines, one passing through points A and B, the other passing through points A and C. And let there be another movable line passing through points $D.E.$ This one, moved over those two lines, and touching on AB at R, AC at $S.$, will give the straight line passing through $R.S.$, or determined from the points R and S. And since all the points such as R and S are determined from the points $A.B.C.$, and the lines connecting them are determined from them, and all points of these

[1]The manuscript has $C.B.\infty C.B.$.

plani per motum rectae) sint omnia puncta plani; patet omnia puncta plani ex tribus punctis $A.\,B.\,C.$ esse determinata.

At sumtis quatuor punctis determinatum est spatium ipsum infinitum, seu locus omnium punctorum quorum unumquodque est determinatum dato ejus situ ad quatuor puncta data; est spatium infinitum, seu locus omnium punctorum in universum, seu quod idem est omnis puncti locus est determinatus, si datus sit ejus situs ad quatuor puncta data, sed hoc debet demonstrari.



[*Fig. 10*]

Cum supra dictum sit punctum C durante motu eodem modo se habere ad rectam AB ergo recta AB eodem modo se habebit ad tria puncta $1C.2C.3C$. ergo etiam eodem modo se habebit ad planum per haec tria puncta determinatum. Et eodem modo cum puncta tria planum determinantia eudem habeant situm ad duo quaedam puncta A et B, etiam ipsum planum sit determinatum eundem ad ea situm habebit.

Demonstrandum est easdem propositiones esse reciprocas, seu omnia puncta eundem simul situm ad duo diversa puncta habentia, cadere in planum; et ad tria habentia cadere in rectam. His habitis conjungi poterunt demonstrata alibi de congruis, et demonstrata hic de determinatis, et habebitur[,] puncta duo eundem situm habentia ad quatuor puncta data, quae eundem situm ad tria aliqua puncta non habent, congruere.

lines (from the proposed generation of the plane by the motion of lines) are all the points of the plane, it is clear that all the points of the plane are determined from these three points $A.B.C$.

And infinite space itself is determined by taking four points; or i.e. the locus of all points each of which is determined given its situs to four given points is infinite space, or the locus of all points in general; or what is the same thing, the locus of every point is determined if its situs to four given points is given; but this must be demonstrated.

[Fig. 10]

As it was said above that the point C during its motion relates in the same way to the line AB, therefore the line AB will relate in the same way to the three points $1C.2C.3C$. Therefore it will also relate in the same way to the plane determined by these three points. And in the same way, when three points determining a plane have the same situs to some two points A and B, the determined plane itself will also have the same situs to them.

It should be demonstrated that the same propositions are reciprocal, or that all points having simultaneously the same situs to two distinct points fall on a plane, and [those] having [the same situs] to three fall on a line. With these things obtained, one may join those demonstrated elsewhere about congruents, and those demonstrated here about things determined, and one will obtain that two points, having the same situs to four given points which do not have the same situs to three other points, will agree.