Circa Geometrica Generalia

G.W. Leibniz, 1678-80?
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DRAFT, February 2025

[Manuscript]
[Typescript]

Concerning General Geometrics and Calculus Situs or depiction with characters[1]
Miscellaneous Observations preliminary to establishing a completely new Geometric Analysis

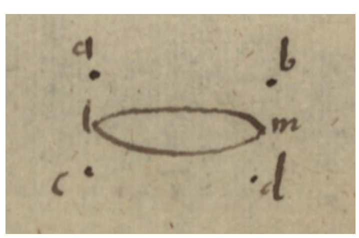
(1) The point is the simplest of all those which are in extension. Hence:
((1)) Point is similar to point, $a \sim b$
(2) Point is equal to point $a = b$
(3) Point is congruent to point $a \simeq b$
These will have use for demonstrating similarities, equalities or congruences of other things that are determined by certain points. Add §60 below.

(4) A point coincides with a point which it is assumed to be in, or if b is in a then $a \infty b$. To these paragraphs 1, 2, 3, 4, add §60 below.

((4)) In fact, in general whatever is situated in a point coincides with the point itself. If multiple points have some common property, and so each one of them is called by the common name X, then the locus common to all and proper to them alone we will call $\overline{X}$. Or $\overline{X}$ will signify:

(5) every point X is in $\overline{X}$, and
(6) every point in $\overline{X}$ is X.
(7) If every X is Y, then $\overline{X}$ *will be in* $\overline{Y}$.
(8) If $\overline{X}$ is in $\overline{Y}$ every X will be Y.
(9) If $\overline{X}$ is in $\overline{Y}$ and $\overline{Y}$ is in $\overline{X}$, then $\overline{X}$ and $\overline{Y}$ will *coincide.*
(10) If $\overline{X}$ and $\overline{Y}$ coincide, $\overline{X}$ will be in $\overline{Y}$ and $\overline{Y}$ will be in $\overline{X}$.
(11) If A is in $\overline{X}$ and $\overline{X}$ in $\overline{Y}$, A will be in $\overline{Y}$. This is demonstrated thus. If A is in $\overline{X}$, of course A is X (by article 6). Now since $\overline{X}$ is in $\overline{Y}$ by hypothesis, every X will be Y (by 8). Therefore (by common Logic) A will also be Y. Therefore (by 5) A will be in $\overline{Y}$. Quod erat demonstrandum. This proposition could be stated thus: what contains the containing contains the contained.

(12) If the situs of points a and b to each other is the same as the situs of points c and d, then points l and m of a rigid body which can be attached to a and b can also be attached to c and d.



[Fig. 1]

[1]Leibniz deleted an alternate title: *On depiction with characters; or, on the Representation of figures through symbols; miscellaneous observations and General Geometrics*

(13) And conversely, if this can be done, the situs of the points will be the same.

(14) The situs of the point a to b is the same as that of the point b to a.

(15) If points l and m of a rigid body can be attached to points a and b, and indeed l to a and m to b, then also, in reverse, l can be attached to b and m to a. For the situs of the point a to the point b is the same as that of the point b to the point a (by 14). Therefore (by 12) what was stated can be done.

(16) If [certain] points of a determined rigid body are determined, which can simultaneously touch given points, the situs of the given points among themselves will be determined. On things determined, see also §25.65.

(17) $a.b$ signifies the situs of points a and b between themselves.[2] And $a.b.c$ signifies the situs of the three points a and b and c among themselves.

(18) If $a.b.c$ is given, $a.b$ is given.

(19) If $a.b$ and $a.c$ and $b.c$ are given, $a.b.c$ is given.

(20) $a.b \simeq c.d$ signifies that the situs between points a and b is the same as that between points c and d, or some rigid thing whose extremes are a and b can be understood as congruent to a rigid thing whose extremes are c and d. Or the points a and b can be brought to agree [posse congruere] with the points c and d while preserving the situs that a and b have between themselves.

(21) If $a.b \simeq l.m$, and $a.c \simeq l.n$ and $b.c \simeq m.n$, then $a.b.c \sim l.m.n$.

(22) If $a.b.c \simeq l.m.n$ then $a.b \simeq l.m$ and so on, pairs to corresponding pairs.

(23) If $a.b.c \simeq l.m.n$ and $a.b.d \simeq l.m.p$ and $a.c.d \simeq l.n.p$ and $b.c.d \simeq m.n.p$[3] then $a.b.c.d \simeq l.m.n.p$.

(24) If there are[4] points of two extensions having some common nature, which are of sufficient number for determining this nature down to a specific individual, and these points have the same situs among themselves in the one as the same number have in the other, [then] these two extensions will be congruent to each other. Let the common nature be $\odot$ and suppose that with four points determined in $\odot$, an individual of $\odot$ is determined, or there is only a unique $\odot$ that has the same four points, and let there be two things F and G of which F is $\odot$ and G is $\odot$ as well, and let four points be assumed in F such as a, b, c, d, and likewise four points in G such as l, m, n, p, and let $a.b.c.d \simeq l.m.n.p$; then $F \simeq G$. [Or expressing it more briefly, let $\overline{a.b.c.d.F}|$-un. (that is, a unique F from the given $a.b.c.d$, and is of F to $a.b.c.d$) and at the same time $\overline{l.m.n.p.G}|$-un., and let $a.b.c.d \simeq l.m.n.p$, then $F \simeq G$.[5]] For example, if there are two Elliptical circumferences and four points on one are situated among themselves in the same way as four points on the other, then these two Elliptical circumferences will be congruent. Since, with four points being given, an ellipse is given. On determination, see also §65 below.

(25) Things are *similar*[6] which, considered separately, cannot be distinguished, or in which, considered in themselves, no differentiating attribute can be noted, but one needs either to compare both with each other, or to compare some third thing with both. Thus if two figures are similar, no proposition (that assumes nothing from outside) can be articulated about one that cannot also be articulated about the other. So if an eye is stationed successively in two rooms made from the same material, if they are dissimilar, it will note some difference, in situs and order, or even in the proportions of parts or lines to each other, and in the angles compared with right [angles]. But if no such thing can be noted, then the eye will have nothing with which to distinguish the one from the other, unless either it views both simultaneously from outside and compares them, or it brings with it some measure (the limbs are such a natural measure in humans; in fact even the back [fundus] of the eye, if there is a notable difference in magnitude). Hence, e.g., two circles are similar; for examine each separately, draw whatever lines you like, consider the ratios of angles to right [angles] and the ratios of straight lines to each other; you will note nothing in one that you wouldn't note in the other. But if you compare two Ellipses, you will easily note a difference. For draw a line out from the center to the circumference, assuming some angle to the axis, and note the ratio of that line to the axis of the Ellipse; do the same in the other Ellipse at the same angle; frequently you will obtain a different ratio, and thus easily distinguish one from the other.

[2] Leibniz wrote $a.b.$ rather than $a.b$ here, and in general was inconsistent about whether or not to include the period at the end of these expressions. We have chosen to exclude the final periods.

[3] Leibniz duplicated $a.c.d \simeq l.n.p$; we changed the second instance in order to complete the pattern.

[4] Leibniz has multiple edits and insertions in this sentence. We insert a 'sint' to create a fully formed clause.

[5] The bracketed portion was added in the margin and later deleted. It appears to be partially ungrammatical.

[6] Italics added.

(26) If the determiners are similar, and the mode of determination itself is similar, the things determined will also be similar. About determination, add §65.75 below.

(27) Hence equiangular triangles are similar, for, given one side and two (and so also three) angles, the triangle is determined; if therefore the angles are the same in both, since a side is similar to a side, namely a line to a line, nothing appears in the determiners from which an attribute could be educed for one that could not also be educed for the other.

(28) Now similar triangles have proportional sides, otherwise some proportion of the sides could be noted in one that could not be noted in the other. Therefore, by the preceding, equiangular triangles have proportional sides.

(29) Conversely, triangles whose sides are proportional are equiangular. For, given three sides, a triangle is determined; if the sides are already proportional, no differentiating attribute can be found in the determiners, namely the sides. Therefore they are similar; therefore, in both, there is the same ratio of angles of the same triangle to each other as well as to their sum; but the sum of the angles is the same in both (since it makes two right angles) therefore also the angles (having the same ratio to this sum in both, otherwise a difference could be noted) will be the same in both.

(30) Things that are similar according to one mode of determining are also similar with respect to another mode of determining. Thus, if two triangles are similar with respect to sides, or have the same ratio in both of individual sides to the sum of sides, they will also be similar with respect to angles, or will have the same ratio in both of individual angles to the sum of angles.

(31) Things are *homogeneous* which either are similar or can be rendered similar by transformation, such as a straight and a circular curve, a gibbous and a planar surface. Indeed since every curve can be extended to a straight line, every surface flattened out and converted into a square, every solid converted into a cube, and a line is similar to a line, a square to a square, a cube to a cube, it is clear that all curves, surfaces, and solids are homogeneous with each other. In fact the definition of Homogenea that Euclid uses cannot be adapted here, since one cannot find even the smallest congruent portion, and neither, therefore, any approximating common measure, however exact. They also seem to be comparable from the generating cause, for if two points move with equal speed and time, the curves described, though dissimilar, will still be equal; but if the speed is the same and the time unequal, they will be [in ratio] as the times, and thus homogenea will be things of which there is a ratio. However, the Euclidean definition could be adapted here as well if the curved [lines] and gibbous [surfaces] were considered as polygons or polyhedra of infinitely many angles. See also §38 below.

((31)) A transformation is a change which is so made that the simplest things that are-in both are the same. Indeed, although sometimes parts remain, as when a square is changed into a right isosceles triangle, still sometimes no part remains, but only the points, as when a circle is changed into an equal square.

(32) *Equals* are things which either are congruent or can be rendered congruent by transformation.

(33) A *greater* is that whose part is equal to another whole (the lesser).

(34) A *lesser* is that which is equal to the part of another (the greater).

(35) Hence it is demonstrated that a part is less than the whole, or the whole is greater than a part. For the part is equal to a part of the whole (namely itself), therefore less than the whole.

(36) If $A \ulcorner B$, then $B \urcorner A$.

(37) If something is neither greater nor lesser, but nonetheless homogeneous, it will be equal. Indeed since it is homogeneous, it can be rendered similar, therefore let it be similar, and since all similar things can be understood to be made from each other by continuous increment or decrement, or to have a common generation, of course that which is generated earlier in the increasing (the decreasing) will be the lesser (the greater). But those that are generated at the same time will be *equals*, which proposition can be regarded as a new definition of equality. The same can be demonstrated also from the definition above; when two proposed things are similar, let them be applied to each other, the corresponding [elements] to the corresponding [elements], then either they will be congruent and equal, or one will exceed [the other] everywhere, otherwise they would not be similar; indeed, if the one does not exceed everywhere, their boundaries will somewhere intersect each other and somewhere not intersect, which is absurd, for only corresponding things should coincide. But these things, if needed, can demonstrated more carefully. Briefly, things which are similar cannot be distinguished except by magnitude. Hence I conclude: if A not $\ulcorner B$ and A not $\urcorner B$ and A Homog. B, then $A = B$.

(38) If B is in A and both are homogeneous but do not coincide, then A is the *whole* and B the *part*.

But homogeneous things must be defined, as we have done above in §31, so that their concept does not presuppose the whole and part, of course, otherwise it makes a circle.

((38)) I call *incommunicant* those *parts* of the same whole which have no part in common, and *communicant* those which do have [a part in common].

(39) The whole and the sum of all incommunicant parts are equal to each other. Indeed, by conjoining these parts, the whole arises [inde fit], or by dividing the whole, these parts arise. Therefore they can be made to coincide and therefore all the more to be congruent (for all coincidents are all the more congruent, or each thing is congruent to itself); but things which can be made congruent are equal.

(40) Two coincident things are congruent, or each thing is congruent to itself. Or in notation, if $A \infty B$, then $A \simeq B$.

(41) Things which are congruent are also equal. If $A \simeq B$, then $A = B$.

(42) Things which are congruent are also similar. If $A \simeq B$, then $A \sim B$.

(43) Things which are simultaneously similar and equal are congruent. If $A \sim B$ and $A = B$, then $A \simeq B$.

((43)) I define *congruent* things as those which cannot be distinguished even when brought together, unless something from outside be assumed, as two equal and similar eggs cannot be distinguished except by their situs to external things. From this of course follow §41 and §43. But §43 is proved thus: things which are equal are congruent or can be rendered so by transformation §32, but things which are also similar have no need of transformation.

(44) Things which are similar are Homogeneous, or if $A \sim B$, then A Homog. B. This is clear from §31.

(45) *Distance* is the magnitude of the shortest curve from one thing to another, as the distance of two points is a line, and that of a point from a line is the perpendicular. I express it thus: AB.

(46) If points A and B are more distant than points C and D, then on any curve drawn from A to B a point can be taken whose situs to A (or to B) is the same as the situs of C to D. What situs, see above §12. This could be stated thus: If $AB \ulcorner CD$ and there is a curve AXB, there will be some point E such that E is X and $AE \simeq CD$. This can be demonstrated. Since, while tending from a point to a point, one cannot reach a greater distance except through a lesser. Indeed, in general:

(47) In every continuous change, from a lesser variation one reaches a greater through all intermediates.

(48) Every extensum which is partly inside and partly outside another intersects its extremity, for somewhere it begins to be inside it, whereas it had been outside just before.

(49) For the extremity or boundary is understood to be *intersected* by some extensum if two points in the extensum can be taken, separated by as small an interval as one likes from the common concurrence, of which one falls outside the boundary, the other inside.

(50) Something *is tangent* which, as it tends toward something, when it reaches it, recedes from it again. And so what is tangent can equally be compared to something intersecting twice, which, when it has entered, again exits, and hence it intersects upon entering as well as upon exiting; but the moments of entry and exit are understood to coincide in the contact, and the portion immersed inside the boundary is regarded as infinitely small.

(51) Every curve that returns to itself cuts off a part from the surface on which it is drawn, or divides the surface into two parts, so that from a point placed in one part no curve can be drawn to a point placed in the other without intersecting that curve. Of course if some part of an extensum is taken, and a division or separation from a remainder that is incommunicant (or having no common part, but only a common edge [terminus]) is established in the common edge itself, it is necessary that the separator return to the point from which it began, because the initial point of the separation ends the cohesion and begins the separation, whereas the final point of the separation ends the separation and begins the cohesion or that which is to be separated; until of course the initial and final point of the separation coincide.

(52) Every whole surface of a finite body encloses it such that no curve can be drawn from a point outside the body to a point inside the body without intersecting that surface.

(53) A curve is the path of a point, so if a moveable point is X, its successive locus is the curve $\overline{X}$.

(54) A surface will be the path of a curve $\overline{X}$ or $L\overline{X}M$ not incident upon its previous tracks; it can be denoted by $\overline{\overline{X}}$ or by $\overline{L\overline{X}M}$.

(55) A body is the path of a surface not incident upon its previous tracks; it can be denoted by $\overline{\overline{\overline{X}}}$ or by $\overline{\overline{L\overline{X}M}}$.

(56) A body cannot move without being incident upon its previous tracks, and so there does not exist another dimension above the curve, surface, and body; of course [that is] in extension, since if besides mass one adds powers, one can ascend to infinity, which however does not vary at all in extension, nor does it produce new figures.

((56)) An extensum is that in which things having situs can be taken, indefinite in number.

(((56))) It is the nature of *situs* that all things which have situs to others also have situs among themselves.

(57) A *point* is the edge of a curve.

(58) A *curve* is the edge of a surface.

(59) A *surface* is the edge of a body.

(60) A *point* is the minimum of things which are in extension, or have situs; or i.e. that which has situs but does not have extension. See §1, 2, 3, 4.

(61) *Space* is that in which, viewed in itself, nothing else besides extension can be considered, such as the locus that remains inside a vessel when water is taken away and substituted with wine.

((61)) Space continues to infinity, for indeed no reason can be provided for limitations, since it is everywhere uniform. But general space, or the locus of all things, is nothing other than a pure absolute extensum, or a maximal extensum, as a point is a minimal one.

(62) All points are in the same space. Or, a body can be given comprehending however many given points.

((62)) Any points whatever have a situs among themselves.

(63) A curve can be drawn from any point to any point.

((63)) Through however many points, finite in number, of the same continuous body, a curve can be drawn that does not leave that body.

(64) A curve can be drawn passing through however many given points and avoiding however many given points. Which I demonstrate thus. Let there be a body containing at once all the given points, those to be touched as well as those to be avoided; let parts containing the points to be avoided be removed from it, small enough that the remaining points which are to be preserved or touched are not harmed, or removed at the same time; since therefore with these parts removed the body nonetheless remains continuous, therefore (from §63) a curve can be drawn in it touching all the remaining points, while remaining in that body, and thus avoiding the ones removed from the body. Which was to be achieved.

(65) Something is *determined* which is entirely unique given certain of its posited conditions (see §16, 24, 26). For example, a point A will be said to be determined from the given situses $A.B$ (or of A to B), $A.C$, $A.D$, $A.E$, if it is impossible for another point to be given that has the same situs to the points $B.C.D.E$. That is, if supposing $A.B.C.D.E \simeq F.B.C.D.E$ one has $A \infty F$, A will be determined from them. This can be expressed: A determ. by $A.B.C.D.E$. Thus a circle is determined by the plane and the center being given in position and the radius in magnitude.

(66) If a point A is determined by its situs to some other points, such as B, C, D, E, then any one of them, such as B, will be determined similarly by its situs to the points A, C, D, E. Or, if some points have such a relation among themselves that one is determined by its situs to the remaining ones, any other will also be determined by its situs to the ones remaining besides itself.

(67) Hence it suffices to write the determining relation of points as $\overline{A.B.C.D.E}|$-un., i.e. that this relation is unique. Which signifies that each of these is determined from its situs to the remaining ones, or if $A.B.C.D.E \simeq F.B.C.D.E$ is supposed, then $A \infty F$ holds, and also if $A.B.C.D.E \simeq A.G.C.D.E$ is supposed, then $B \infty G$ will hold, and so on for C, D, E the same thing will take place.

(68) If the determiners are congruent, the determined will also be congruent, the mode of determining being the same. See article 24 above, for example two radii of circles, ellipses, generating [by rotation about ...[7]] two circles, spheres, spheroids.

(69) In fact if the mode of determining is the same and the determiners are equal, the determined will also be equal; thus a cylindrical surface will be equal to a rectangle of the same altitude as the cylinder, if the base of the rectangle is equal to the circumference of the circle generating the cylinder. For in the same way that the rectangle is generated by drawing a line along the altitude, the cylindrical surface is generated by drawing the circumference of the circle along the same altitude.

[7] The MS shows two abandoned attempts to explain the generation of circles and ellipses by rotation.

(70) It is false that the determined are proportional to the determiners, even if the mode of determination is the same, unless the determined are homogeneous to the determiners. Otherwise, it would follow that circles are to each other as [their] radii, for, given a center and radius, a circle is determined.

(71) On the contrary, that circles are as the squares of [their] diameters, which Euclid showed with many detours in book 10 of the *Elements*, is immediately clear to me at the first glance from the definition of similarity. For circle A with circumscribed square B comprises a similar figure to circle C with circumscribed square D. For a circle is similar to a circle, a square similar to a square, and the method of applying the circle to the square is also similar in both cases. *Therefore, the ratio of A to B is the same as that of C to D* (otherwise something could be observed in $A.B$ viewed in itself distinguishing it from $C.D$ viewed in itself; indeed subtracting A from B, and the remainder from A as many times as possible, and again the second remainder from the first remainder as many times as possible, and so on, a difference in the number of possible subtractions would be observed when working with figure $A.B$ from that which would turn out when working with figure $C.D$). Therefore, by inverting, *the ratio of A to C will also be the same as that of B to D*. Quod erat demonstrandum. By the same method one demonstrates that

(72) *all surfaces are as the squares of the determining lines*, and similarly,

(73) spheres or other *similar solid figures are as the cubes of the determining lines.* But one may not say that circles A and C are as their diameters E and F, even if the circles with their diameters also comprise similar figures in both cases, for, since there does not exist a ratio of a circle (a surface) to a diameter (a curve) because they are not homogeneous, one cannot say A is to E as C to F, nor therefore by inverting that A is to C as E to F.

(74) If the determiners are similar and the mode of determining is the same, the determined will also be similar (see above §26). Hence, all circles are similar among themselves, likewise all squares, and parabola to parabola, and an Ellipse is similar to an ellipse when the right and transverse sides [are] proportional. But a curve parallel to an Ellipse is not an Ellipse, and a curve parallel to a parabola is not a parabola. What exactly these parallel curves are, we will say in its own place.

(75) If the determiners are coincident and the mode of determining is the same, the determined will also be coincident. Thus if a plane and a center on it are given in position, and a radius in magnitude, a circle is determined. If therefore there are said to be two circles in the same plane, or else in planes thought to be two but in reality coincident, whose radii are equal, and it is found that their centers coincide, then the circles themselves will coincide.

(76) If A is similar, equal, congruent, coincident to B, and B to C, [then] A will also be to C.

(77) Equals can be substituted in the place of equals with equality preserved, that is, if you add or take away equals, or multiply or divide equals by equals, equals result. But it does not follow, nor can it be demonstrated from this axiom of ours, that whatever things produce equals when multiplied by themselves are equal to each other, for $+3$ and -3 each produce 9 when multiplied by themselves, but they are not equal since their difference is 6; therefore when powers are equal the roots are not equal, even though when roots are equal the powers are equal.

(78) Coincidents can be substituted for those with which they coincide, preserving everything, indeed they are actually the same, and I define as *the same* those that can be substituted everywhere with truth preserved, in propositions, of course, that are direct and not reflected in the same mode of consideration. An arc of a circle and a uniform curve in a plane can be substituted everywhere for each other except reflexive propositions, such as it is if someone said: an arc of a circle can be conceived without any respect to a plane; although if one wanted to proceed more rigorously, this substitution could be defended even in reflexive [propositions].

(79) If B is A, and C is A, and in fact B and C coincide, or $B \infty C$, it is said to be *one A*.

(80) If B is A, and C is A, and B is not C, and C is not B, they are said to be *two A*. And if B is A, and C is A and D is A, and B is not C nor D, and C is neither B nor D, and D is not B nor C, they are said to be *three A*. And so on. And in general when not just one is A, they are said to be *many.* And this is the origin of *Numbers*; and this very expression is seen in the Athanasian Creed, although its use there seems to contradict this definition, but the contradiction is removed by a distinction.

Notes