De Analysi Situs

G.W. Leibniz, 1704-06?
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What is commonly promoted as *mathematical analysis* is analysis of magnitude, not of situation, and so it pertains directly and immediately to arithmetic, but is applied to geometry in a roundabout fashion. Whence it happens that many things easily become clear from consideration of situation which the algebraic calculus shows with difficulty. To reduce geometric problems to algebra, that is, to reduce what is determined by figures to equations, is often a rather lengthy process; and yet another protracted and arduous effort is needed to return from equation to construction, from algebra to geometry; and often by this route we get constructions that are not quite suitable, unless we are fortunate enough to strike upon certain unforeseen suppositions or assumptions. Descartes himself tacitly admits this while solving a certain problem of Pappus in his third book of geometry. And of course algebra, whether of numbers or symbols, adds, subtracts, multiplies, divides, and extracts roots, which is just arithmetic. Indeed, logistic itself, or the science of magnitude and proportion in general, treats nothing but general or indeterminate number and those symbols of operations on it, because magnitude is actually estimated by the multiplicity of determinate parts, which nevertheless varies when one or another measure or unity is assumed while the thing remains the same. Hence it is not surprising that the science of magnitudes in general is a kind of arithmetic, since it deals with uncertain numbers.

The ancients had another kind of analysis, distinct from algebra, which comes nearer to the consideration of situation, treating the givens and the sites or loci of the things sought. And Euclid's little book on givens, on which we have the commentary of Marinus, aims at this. Now, loci that are planar, solid, or linear were dealt with by others, then by Apollonius, whose propositions were preserved by Pappus, from which more recently some have recovered the planar and solid loci, but so that they appeared to show the truth rather than the source of the ancient doctrine. This kind of analysis, however, neither reduces the matter to calculus, nor indeed is it carried all the way to the first principles and elements of situation, which is necessary for a perfect analysis.

Therefore, the true analysis of situation is yet to be supplied, and this is evident from the fact that all analysts either practise algebra in the new fashion or treat the givens and things sought according to the ancient form, and they must assume many things from elementary geometry which are not deduced from consideration of magnitude but of figure, and neither are they clear by any determinate path, until now. Euclid himself was compelled to assume certain quite obscure axioms without proof, that the rest might proceed. And the demonstration of theorems and the solution of problems in the *Elements* sometimes appears to be a work more of toil than of method and art, although now and then the artistry of the process seems to be concealed.

Figure in general includes, besides quantity, also quality or form; and just as equals are those whose magnitude is the same, so similars are those whose form is the same. And indeed, the consideration of similarities and forms is clearly broader than mathesis and is learned from metaphysics, though it does also have multiple uses in mathesis and is beneficial for the algebraic calculus itself; but similarity is regarded most of all in situations, or figures of geometry. And so a truly geometric analysis regards not only equalities and proportions, which are actually reduced to equalities, but also similarities, and should employ the congruences arising from equality and similarity combined.

Now the reason why geometers have not made satisfactory use of the consideration of similarity I judge to be this, that they had no general notion of it sufficiently distinct or adapted to mathematical inquiries, a fault of philosophers, who are customarily content with vague definitions equal in obscurity to what is defined, especially in first philosophy; whence it is no wonder that that doctrine tends to be sterile and verbose. And so it is not enough to say that similars are those whose form is the same unless one has in turn a general notion of form. Having formulated an explanation of quality or form, however, I discovered that in the end the matter comes down to this, that similars are those which cannot be distinguished when observed individually. For quantity can only be apprehended by the co-presence of things, or by an actual intervening attachment of them. Quality presents something to the mind which you recognize in a thing separately and can apply to compare two things, even when no actual attachment intervenes by which one thing is compared with the other either immediately or by mediation of a third thing as a measure. Let us imagine that two temples or buildings have been constructed by the rule that nothing can be apprehended in the one that you would not observe in the other: namely the material is everywhere the same, white marble from Paros if you like; the proportions of all the walls, the columns, and the rest are the same in both; the angles in both are the same, or of the same ratio to a right angle; and so whoever is led into these two temples with eyes closed, and after entering opens them, and goes about now in the one and now in the other, will not find any token in them by which to distinguish the one from the other. And yet they can differ in magnitude, and so can be distinguished if they are viewed simultaneously from the same place, or even (allowing them to be far from each other) if some third thing is transported and compared now with the one and now with the other, just as some measure such as a cubit or foot or something else suitable for measuring is applied now to the one and now to the other, for then at last a basis for distinguishing will be given by the observed inequality. It is the same if the viewer's body itself, or a limb, which certainly goes with oneself from place to place and serves the role of a measure, is applied to these temples; for then the different magnitude, and through it a method of distinguishing, will become apparent. But if the viewer is considered to be nothing but a sighted mind, as though constituting a point, not carrying with it any magnitudes either in reality or in the imagination, and considering in things only that which may be followed by the intellect, like numbers, proportions, and angles, then no distinction will present itself. Therefore, these temples will be called similar, since they could not be distinguished except by this co-observation, either among themselves or with a third thing, and not at all when viewed individually and in themselves.

This perspicuous and practical and general description of similarity will be profitable to us for geometric demonstrations, as will shortly become clear. Indeed, we will say that two proposed figures are similar if nothing could be noted in one, viewed individually, that could not equally be apprehended in the other. And so it follows that the ratio or proportion of the ingredients must be the same in both, otherwise a distinction would appear in themselves individually, i.e. even if no co-observation of them both is arranged. But geometers, since they lacked a general notion of similarity, defined similar figures from equal corresponding angles, which is specific and does not reveal the nature of similarity itself in general. And so they had to take a circuitous route to demonstrate what is clear at the first glance from our notion. But let us come to examples.

It is shown in the *Elements* that similar or equiangular triangles have proportional sides, and conversely, but Euclid finally completes this in the fifth book by many detours, when he could have shown it immediately in the first Element, if he had followed our notion. We will show first that equiangular triangles are similar.

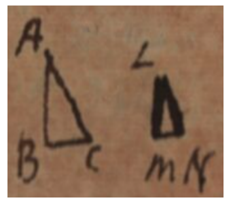

Figure 1

Let there be a triangle ABC and again another LMN, and let the angles A, B, C be equal to L, M, N respectively; I claim that the triangles are similar. But I use this new axiom: Whatever cannot be distinguished through the determiners (i.e. sufficient givens) cannot be distinguished at all, since everything else arises from the determiners. Now given the base BC and given the angles B and C (and so also the angle A), the triangle ABC is given; likewise, given the base MN, and given the angles M, N (and so also the angle L), the triangle LMN is given. But the triangles cannot be distinguished through these sufficient givens individually. Indeed, in each one the base is given, and the two angles to the base; now the base cannot be compared with the angles, therefore, nothing else remains that could be examined from the determiners in either triangle viewed individually, other than the ratio of each given angle to a right angle or two right angles, that is, the magnitude of the angle itself. Since these things are found to be the same, it is necessary that the triangles cannot be distinguished individually, and so are similar. Indeed, I might add as a sort of scholium that even if the triangles can be distinguished in magnitude, nonetheless the magnitude cannot be recognized except through co-observation either of both triangles simultaneously, or of each one with some measure, but then they would not be viewed just individually, which was required.

Conversely, it is clear that similar triangles are also equiangular; otherwise, if there were some angle, say A, in triangle ABC, for which no equal angle could be found in triangle LMN, certainly there would be an angle in

ABC having a ratio to two right angles (or to the sum of all angles of the triangle) which none has in *LMN*, and this suffices to distinguish individually triangle *ABC* from triangle *LMN*. It is evident as well that similar triangles have proportional sides. For if some two sides were given, such as *AB* and *BC*, having a ratio to each other that no sides of triangle *LMN* have to each other, then the one triangle could be distinguished individually from the other. Finally, if the sides are proportional, the triangles will be similar. Indeed, since the triangles are given when the sides are given, it suffices (by our axiom) that a distinction could not be obtained from the ratio of the sides, in order for us to judge that it could not be obtained from anything else in these triangles viewed individually. And from these [facts] it is of course also clear that equiangular triangles have proportional sides, and conversely.

In the same way, immediately upon the first glance of the mind, directly from our notion of similarity, it is shown that circles are as the squares of the diameters, which Euclid finally shows in the tenth book and indeed by means of inscribed and circumscribed figures, reducing the matter to absurdity, when actually no detours were necessary.

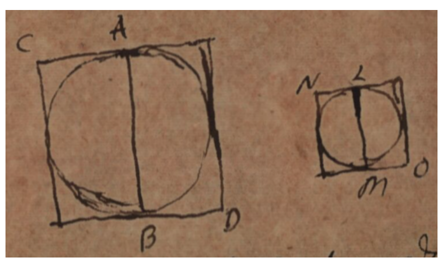

Figure 2

Let a circle be described with diameter *AB*, and let the square *CD*[1] of the diameter be circumscribed on it; and in the same way let a circle be described with diameter *LM*, and let the square *NO* of the diameter be circumscribed on it. The determination of both is similar, the circle to the circle, the square to the square, and the fitting of the square to the circle, and so (by the aforementioned axiom) the figures *ABCD* and *LMNO* are similar. Therefore (by the definition of similarity) the circle *AB* is to the square *CD* as the circle *LM* to the square *NO*, therefore also the circle *AB* to the circle *LM* is as the square *CD* to the square *NO*, as was asserted. By comparable reasoning, spheres

[1] Meaning: the square whose side is the diameter *CD*.

will be shown to be as the cubes of the diameters. And in similars in general, homologous curves, surfaces, and solids will be as the lengths, squares, and cubes of homologous sides, respectively. Something which, until now, has generally been assumed rather than demonstrated.

This consideration, furthermore, which affords such ease in demonstrating truths that would be difficult to demonstrate by another method, also revealed to us a new kind of calculus, a whole heaven apart from the algebraic calculus, equally new in its symbols and in its use of symbols, or its operations. And so I like to call it the analysis of situation, because it expresses [explicat] situation directly and immediately, so that figures might be depicted in the mind through the symbols even without being drawn, and whatever the empirical imagination understands from figures, the calculus will derive from the symbols by sure demonstration, and even obtain everything else which the power of imagining cannot reach. Therefore a supplement to, and if I may say so the perfection of, the imagination is contained in this calculus of situation that I have proposed, and it will have uses never known until now, not only in geometry, but also for the invention of machines and the descriptions themselves of the nature of machines.[2]

[2] The MS has "machinarum naturae" which could be interpreted either as "of the machines of nature" or as "of the nature of machines." A prior, crossed-out draft had "explicationesque" instead of the longer phrase "ipsasque machinarum naturae descriptiones" and thus made no mention of nature.

Notes