W14 - Examples

Polar curves

Converting to polar: π -correction

Compute the polar coordinates of
$$\left(-\frac{1}{2}, +\frac{\sqrt{3}}{2}\right)$$
 and of $\left(+\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.

Solution

For $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ we observe first that it lies in Quadrant II.

Next compute:

$$an^{-1}\left(rac{\sqrt{3}/2}{-1/2}
ight) \quad \gg \gg \quad an^{-1}\left(-\sqrt{3}
ight) \quad \gg \gg \quad -\pi/3$$

This angle is in Quadrant IV. We $add \pi$ to get the polar angle in Quadrant II:

$$heta=\pi-\pi/3$$
 \gg $2\pi/3$

The radius is of course 1 since this point lies on the unit circle. Therefore polar coordinates are $(r, \theta) = (1, 2\pi/3)$.

For $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ we observe first that it lies in Quadrant IV.

Next compute:

$$\tan^{-1}\left(\frac{+\sqrt{2}/2}{-\sqrt{2}/2}\right) \quad \gg \gg \quad \tan^{-1}(-1) \quad \gg \gg -\pi/4$$

This is the correct angle because Quadrant IV is SAFE. So the point in polar is $(1, -\pi/4)$.

Shifted circle in polar

For example, let's convert a shifted circle to polar. Say we have the Cartesian equation:

$$x^2 + (y - 3)^2 = 9$$

Then to find the polar we substitute $x = r \cos \theta$ and $y = r \sin \theta$ and simplify:

$$x^2 + (y-3)^2 = 9$$

 $\gg \gg r^2 \cos^2 heta + (r \sin heta - 3)^2 = 9$
 $\gg \gg r^2 \cos^2 heta + r^2 \sin^2 heta - 6r \sin heta + 9 = 9$
 $\gg \gg r^2 (\sin^2 heta + \cos^2 heta) - 6r \sin heta = 0$
 $\gg \gg r^2 - 6r \sin heta = 0 \gg r = 6 \sin heta$

So this shifted circle is the polar graph of the polar function $r(\theta) = 6 \sin \theta$.

Calculus with polar curves

Finding vertical tangents to a limaçon

Let us find the vertical tangents to the limaçon (the cardioid) given by $r = 1 + \sin \theta$.

- 1. \equiv Convert to Cartesian parametric.
 - Plug $r(\theta)$ into $x = r \cos \theta$ and $y = r \sin \theta$:

$$r(heta) = 1 + \sin heta \quad \gg \gg \quad (x,y) = ig((1+\sin heta)\cos heta, \; (1+\sin heta)\sin hetaig)$$

2. \equiv Compute x' and y'.

• Derivatives of both coordinates:

 $(x', y') \gg \gg$

$$\Big(\cos heta\cos heta+(1+\sin heta)(-\sin heta),\ \cos heta\sin heta+(1+\sin heta)\cos heta\Big)$$

• Simplify:

$$\gg \gg \qquad \left(\, \cos^2 heta - \sin^2 heta - \sin heta, \, \cos heta \left(1 + 2 \sin heta
ight)
ight)$$

- 3. $\models \exists$ The vertical tangents occur when $x'(\theta) = 0$.
 - Set equation: x' = 0:

$$x'(\theta) = 0 \implies \infty \cos^2 \theta - \sin^2 \theta - \sin \theta = 0$$

• 🛆 Solve equation.

• Convert to *only* $\sin \theta$:

$$\gg \gg (1-\sin^2 heta)-\sin^2 heta-\sin heta=0$$

• Substitute $A = \sin \theta$ and simplify:

$$\gg \gg \quad 1-2A^2-A=0 \quad \gg \gg \quad 2A^2+A-1=0$$

• Solve:

$$egin{aligned} A &= rac{-b \pm \sqrt{b^2 - 4ac}}{2a} &\gg \gg \ rac{-1 \pm \sqrt{1 - 4 \cdot 2 \cdot (-1)}}{2 \cdot 2} &\gg \gg & rac{1}{2}, \, -1 \end{aligned}$$

• Solve for θ :

$$A=\sin heta \quad \gg \gg \quad \sin heta=rac{1}{2},\,-1$$

$$\gg \gg \quad heta = rac{\pi}{6}, \, rac{5\pi}{6} \, (ext{for } 1/2) \quad ext{and} \quad heta = rac{3\pi}{2} \, (ext{for } -1)$$

4. E Compute final points.

• In polar coordinates, the final points are:

$$egin{aligned} &(r, heta)=(1+\sin heta,\, heta)\Big|_{ heta=rac{\pi}{6},\,rac{5\pi}{6},\,rac{3\pi}{2}} \ &\gg\gg\quad \left(rac{3}{2},rac{\pi}{6}
ight),\,\left(rac{3}{2},rac{5\pi}{6}
ight),\,\left(0,rac{3\pi}{2}
ight) \end{aligned}$$

• In Cartesian coordinates:

• For $\theta = \frac{\pi}{6}$:

$$\begin{aligned} (x,y)\Big|_{\theta=\frac{\pi}{6}} & \gg \quad \left((1+\sin\theta)\cos\theta, \, (1+\sin\theta)\sin\theta\right)\Big|_{\theta=\frac{\pi}{6}} \\ & \gg \quad \left(\left(1+\frac{1}{2}\right)\frac{\sqrt{3}}{2}, \, \left(1+\frac{1}{2}\right)\frac{1}{2}\right) \quad \gg \quad \left(\frac{3\sqrt{3}}{4}, \, \frac{3}{4}\right) \end{aligned}$$

• For $\theta = \frac{5\pi}{6}$:

$$\begin{aligned} (x,y)\Big|_{\theta=\frac{5\pi}{6}} & \gg \qquad \left((1+\sin\theta)\cos\theta, \ (1+\sin\theta)\sin\theta\right)\Big|_{\theta=\frac{5\pi}{6}} \\ & \gg \gg \qquad \left(\left(1+\frac{1}{2}\right)\frac{-\sqrt{3}}{2}, \ \left(1+\frac{1}{2}\right)\frac{1}{2}\right) \quad \gg \gg \quad \left(-\frac{3\sqrt{3}}{4}, \ \frac{3}{4}\right) \end{aligned}$$

• For $\theta = \frac{3\pi}{2}$:

$$egin{aligned} & (x,y) \Big|_{ heta = rac{3\pi}{2}} & \gg \gg & \left((1+\sin heta)\cos heta, \, (1+\sin heta)\sin heta
ight) \Big|_{ heta = rac{3\pi}{2}} \ & \gg \gg & \left((1-1)\cdot 0, \, (1-1)\cdot (-1)
ight) & \gg \gg & (0,0) \end{aligned}$$

5. \triangle Correction: (0, 0) is a cusp.

- The point (0,0) at $\theta = \frac{3\pi}{2}$ is on the cardioid, but the curve is not smooth there, this is a cusp.
- Still, the left- and right-sided tangents exists and are equal, so in a sense we can still say the curve has vertical tangent at $\theta = \frac{3\pi}{2}$.

Length of the inner loop

Consider the limaçon given by $r(\theta) = \frac{1}{2} + \cos \theta$. How long is its inner loop? Set up an integral for this quantity.

Solution

The inner loop is traced by the moving point when $\frac{2\pi}{3} \le \theta \le \frac{4\pi}{3}$. This can be seen from the graph:



Therefore the length of the inner loop is given by this integral:

$$L=\int_{2\pi/3}^{4\pi/3}\sqrt{(-\sin heta)^2+\left(rac{1}{2}+\cos heta
ight)^2}\,d heta$$
 \gg $\int_{2\pi/3}^{4\pi/3}\sqrt{5/4+\cos heta}\,d heta$

Area between circle and limaçon

Find the area of the region enclosed between the circle $r_0(heta)=1$ and the limaçon $r_1(heta)=1+\cos heta.$

Solution

First draw the region:



The two curves intersect at $\theta = \pm \frac{\pi}{2}$. Therefore the area enclosed is given by integrating over $-\frac{\pi}{2} \le \theta \le +\frac{\pi}{2}$:

$$egin{aligned} &A = \int_{a}^{b} rac{1}{2} (r_{1}^{2} - r_{0}^{2}) \, d heta &\gg > \int_{-\pi/2}^{\pi/2} rac{1}{2} \left((1 + \cos heta)^{2} - 1^{2}
ight) d heta \ &\gg > rac{1}{2} \int_{-\pi/2}^{\pi/2} 2\cos heta + \cos^{2} heta \, d heta &\gg > \int_{-\pi/2}^{\pi/2} \cos heta + rac{1}{4} ig(1 + \cos(2 heta) ig) \, d heta \ &\gg > \quad \sin heta + rac{ heta}{4} + rac{1}{8} \sin(2 heta) \Big|_{-\pi/2}^{\pi/2} \gg > 2 + rac{\pi}{4} \end{aligned}$$

Area of small loops

Consider the following polar graph of $r(\theta) = 1 + 2\cos(4\theta)$:



Find the area of the shaded region.

Solution

- 1. \equiv Bounds for one small loop.
 - Lower left loop occurs first.
 - This loop when $1 + 2\cos(4\theta) \le 0$.
 - Solve this:

$$egin{aligned} 1+2\cos(4 heta) &\leq 0 &\gg & \cos(4 heta) \leq -rac{1}{2} \ &\gg & rac{2\pi}{3} \leq 4 heta \leq rac{4\pi}{3} &\gg & rac{\pi}{6} \leq heta \leq rac{\pi}{3} \end{aligned}$$

2. \Rightarrow Area integral.

• Arrange and expand area integral:

$$egin{aligned} A &= 4 \int_lpha^eta rac{1}{2} r(heta)^2 \, d heta &\gg \gg -4 \int_{\pi/6}^{\pi/3} rac{1}{2} ig(1+2\cos(4 heta)ig)^2 \, d heta \ &\gg \gg -2 \int_{\pi/6}^{\pi/3} 1 + 4\cos(4 heta) + 4\cos^2(4 heta) \, d heta \end{aligned}$$

• Simplify integral using power-to-frequency: $\cos^2 A \rightsquigarrow \frac{1}{2}(1 + \cos(2A))$ with $A = 4\theta$:

$$\gg \gg -2\int_{\pi/6}^{\pi/3} 1+4\cos(4 heta)+4\cdot rac{1}{2}ig(1+\cos(8 heta)ig)\,d heta$$

• Compute integral:

$$\gg \gg \quad 6 heta+2\sin(4 heta)+rac{1}{4}\sin(8 heta)\Big|_{\pi/6}^{\pi/3}$$
 $\gg \gg \quad \pi-rac{3\sqrt{3}}{2}$

Overlap area of circles

Compute the area of the overlap between crossing circles. For concreteness, suppose one of the circles is given by $r(\theta) = \sin \theta$ and the other is given by $r(\theta) = \cos \theta$.

Solution

Here is a drawing of the overlap:



- 1. \equiv Notice: total overlap area = 2× area of red region.
- 2. \equiv Bounds: $0 \le \theta \le \frac{\pi}{4}$.
- 3. \Rightarrow Apply area formula for the red region.
 - Area formula applied to $r(\theta) = \sin \theta$:

$$A = \int_{lpha}^{eta} rac{1}{2} r(heta)^2 \, d heta \qquad \gg \gg \qquad \int_{0}^{\pi/4} rac{1}{2} \sin^2 heta \, d heta$$

• Power-to-frequency: $\sin^2 \theta \rightsquigarrow \frac{1}{2} (1 - \cos(2\theta))$:

$$\gg \gg \int_0^{\pi/4} rac{1}{4} ig(1-\cos(2 heta)ig) d heta$$

$$\gg \gg - rac{1}{4} heta - rac{1}{8} {
m sin}(2 heta) \Big|_0^{\pi/4} \quad \gg \gg - rac{\pi}{16} - rac{1}{8}$$

• Double the result to include the black region:

$$\gg \gg \frac{\pi}{8} - \frac{1}{4}$$