W14 - Examples

Polar curves

Converting to polar: π-correction

Compute the polar coordinates of $\left(-\frac{1}{2}, +\frac{\sqrt{3}}{2}\right)$ and of $\left(+\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.

Solution

For $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ we observe first that it lies in Quadrant II.

Next compute:

$$\tan^{-1}\left(\frac{\sqrt{3}/2}{-1/2}\right) \quad \gg\gg \quad \tan^{-1}\left(-\sqrt{3}\right) \quad \gg\gg \quad -\pi/3$$

This angle is in Quadrant IV. We *add π* to get the polar angle in Quadrant II:

$$\theta = \pi - \pi/3 \quad \gg\gg \quad 2\pi/3$$

The radius is of course 1 since this point lies on the unit circle. Therefore polar coordinates are $(r, \theta) = (1, 2\pi/3)$.

For $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ we observe first that it lies in Quadrant IV.

Next compute:

$$\tan^{-1}\left(\frac{+\sqrt{2}/2}{-\sqrt{2}/2}\right) \quad \gg\gg \quad \tan^{-1}(-1) \quad \gg\gg \quad -\pi/4$$

This is the correct angle because Quadrant IV is SAFE. So the point in polar is $(1, -\pi/4)$.

Shifted circle in polar

For example, let's convert a shifted circle to polar. Say we have the Cartesian equation:

$$x^2 + (y-3)^2 = 9$$

Then to find the polar we substitute $x = r\cos\theta$ and $y = r\sin\theta$ and simplify:

$$x^2 + (y-3)^2 = 9$$

$$\gg\gg \quad r^2\cos^2\theta + (r\sin\theta - 3)^2 = 9$$

$$\gg\gg \quad r^2\cos^2\theta + r^2\sin^2\theta - 6r\sin\theta + 9 = 9$$

$$\gg\gg \quad r^2(\sin^2\theta + \cos^2\theta) - 6r\sin\theta = 0$$

$$\gg\gg \quad r^2 - 6r\sin\theta = 0 \quad \gg\gg \quad r = 6\sin\theta$$

So this shifted circle *is the polar graph of the polar function* $r(\theta) = 6\sin\theta$.

Calculus with polar curves

Finding vertical tangents to a limaçon

Let us find the vertical tangents to the limaçon (the cardioid) given by $r = 1 + \sin\theta$.

1. Convert to Cartesian parametric.
    - Plug $r(\theta)$ into $x = r\cos\theta$ and $y = r\sin\theta$:

$$r(\theta) = 1 + \sin\theta \quad \gg\gg \quad (x, y) = \Big((1+\sin\theta)\cos\theta,\ (1+\sin\theta)\sin\theta\Big)$$

2. Compute x' and y'.
    - Derivatives of both coordinates:

$$(x',\, y') \quad \gg\gg$$

$$\Big(\cos\theta\cos\theta + (1+\sin\theta)(-\sin\theta),\ \cos\theta\sin\theta + (1+\sin\theta)\cos\theta\Big)$$

    - Simplify:

$$\gg\gg \qquad \Big(\cos^2\theta - \sin^2\theta - \sin\theta,\ \cos\theta\,(1 + 2\sin\theta)\Big)$$

3. The vertical tangents occur when $x'(\theta) = 0$.
    - Set equation: $x' = 0$:

$$x'(\theta) = 0 \quad \gg\gg \quad \cos^2\theta - \sin^2\theta - \sin\theta = 0$$

    - ⚠ Solve equation.
        - Convert to *only* $\sin\theta$:

$$\gg\gg \quad (1 - \sin^2\theta) - \sin^2\theta - \sin\theta = 0$$

        - Substitute $A = \sin\theta$ and simplify:

$$\gg\gg \quad 1 - 2A^2 - A = 0 \quad \gg\gg \quad 2A^2 + A - 1 = 0$$

        - Solve:

$$A = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \gg\gg$$

$$\frac{-1 \pm \sqrt{1 - 4\cdot 2\cdot(-1)}}{2\cdot 2} \quad \gg\gg \quad \frac{1}{2},\ -1$$

        - Solve for θ:

$$A = \sin\theta \quad \gg\gg \quad \sin\theta = \frac{1}{2},\ -1$$

$$\gg\gg \quad \theta = \frac{\pi}{6},\ \frac{5\pi}{6}\ (\text{for } 1/2) \quad \text{and} \quad \theta = \frac{3\pi}{2}\ (\text{for } -1)$$

4. Compute final points.

- In polar coordinates, the final points are:

$$(r,\theta) = (1+\sin\theta,\, \theta)\Big|_{\theta=\frac{\pi}{6},\, \frac{5\pi}{6},\, \frac{3\pi}{2}}$$

$$\gg\gg \quad \left(\frac{3}{2}, \frac{\pi}{6}\right),\ \left(\frac{3}{2}, \frac{5\pi}{6}\right),\ \left(0, \frac{3\pi}{2}\right)$$

- In Cartesian coordinates:
  - For $\theta = \frac{\pi}{6}$:

$$(x,y)\Big|_{\theta=\frac{\pi}{6}} \quad \gg\gg \quad \Big((1+\sin\theta)\cos\theta,\ (1+\sin\theta)\sin\theta\Big)\Big|_{\theta=\frac{\pi}{6}}$$

$$\gg\gg \quad \left(\left(1+\frac{1}{2}\right)\frac{\sqrt{3}}{2},\ \left(1+\frac{1}{2}\right)\frac{1}{2}\right) \quad \gg\gg \quad \left(\frac{3\sqrt{3}}{4}, \frac{3}{4}\right)$$

  - For $\theta = \frac{5\pi}{6}$:

$$(x,y)\Big|_{\theta=\frac{5\pi}{6}} \quad \gg\gg \quad \Big((1+\sin\theta)\cos\theta,\ (1+\sin\theta)\sin\theta\Big)\Big|_{\theta=\frac{5\pi}{6}}$$

$$\gg\gg \quad \left(\left(1+\frac{1}{2}\right)\frac{-\sqrt{3}}{2},\ \left(1+\frac{1}{2}\right)\frac{1}{2}\right) \quad \gg\gg \quad \left(-\frac{3\sqrt{3}}{4}, \frac{3}{4}\right)$$

  - For $\theta = \frac{3\pi}{2}$:

$$(x,y)\Big|_{\theta=\frac{3\pi}{2}} \quad \gg\gg \quad \Big((1+\sin\theta)\cos\theta,\ (1+\sin\theta)\sin\theta\Big)\Big|_{\theta=\frac{3\pi}{2}}$$

$$\gg\gg \quad \Big((1-1)\cdot 0,\ (1-1)\cdot(-1)\Big) \quad \gg\gg \quad (0,0)$$

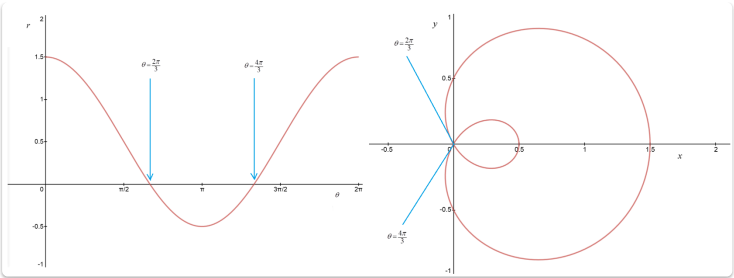
5. ⚠ Correction: $(0,0)$ is a cusp.
   - The point $(0,0)$ at $\theta = \frac{3\pi}{2}$ is on the cardioid, but the curve is not smooth there, this is a cusp.
   - Still, the left- and right-sided tangents exists and are equal, so in a sense we can still say the curve has vertical tangent at $\theta = \frac{3\pi}{2}$.

Length of the inner loop

Consider the limaçon given by $r(\theta) = \frac{1}{2} + \cos\theta$. How long is its inner loop? Set up an integral for this quantity.

Solution

The inner loop is traced by the moving point when $\frac{2\pi}{3} \leq \theta \leq \frac{4\pi}{3}$. This can be seen from the graph:



Therefore the length of the inner loop is given by this integral:

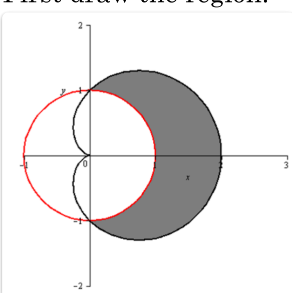
$$L = \int_{2\pi/3}^{4\pi/3} \sqrt{(-\sin\theta)^2 + \left(\frac{1}{2} + \cos\theta\right)^2}\, d\theta \quad \gg\gg \quad \int_{2\pi/3}^{4\pi/3} \sqrt{5/4 + \cos\theta}\, d\theta$$

Area between circle and limaçon

Find the area of the region enclosed between the circle $r_0(\theta) = 1$ and the limaçon $r_1(\theta) = 1 + \cos\theta$.

Solution

First draw the region:



The two curves intersect at $\theta = \pm\frac{\pi}{2}$. Therefore the area enclosed is given by integrating over $-\frac{\pi}{2} \le \theta \le +\frac{\pi}{2}$:

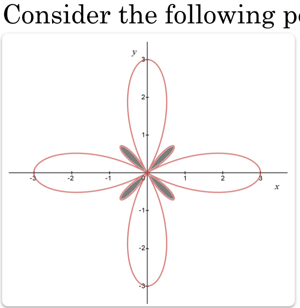
$$A = \int_a^b \frac{1}{2}(r_1^2 - r_0^2)\, d\theta \quad \gg\gg \quad \int_{-\pi/2}^{\pi/2} \frac{1}{2}\left((1+\cos\theta)^2 - 1^2\right) d\theta$$

$$\gg\gg \quad \frac{1}{2}\int_{-\pi/2}^{\pi/2} 2\cos\theta + \cos^2\theta\, d\theta \quad \gg\gg \quad \int_{-\pi/2}^{\pi/2} \cos\theta + \frac{1}{4}\big(1+\cos(2\theta)\big)\, d\theta$$

$$\gg\gg \quad \sin\theta + \frac{\theta}{4} + \frac{1}{8}\sin(2\theta)\Big|_{-\pi/2}^{\pi/2} \quad \gg\gg \quad 2 + \frac{\pi}{4}$$

Area of small loops

Consider the following polar graph of $r(\theta) = 1 + 2\cos(4\theta)$:



Find the area of the shaded region.

Solution

1. Bounds for one small loop.
   - Lower left loop occurs first.
   - This loop when $1+2\cos(4\theta)\leq 0$.
   - Solve this:

$$1+2\cos(4\theta)\leq 0 \qquad \gg\gg \qquad \cos(4\theta)\leq -\frac{1}{2}$$

$$\gg\gg \qquad \frac{2\pi}{3}\leq 4\theta\leq\frac{4\pi}{3} \qquad \gg\gg \qquad \frac{\pi}{6}\leq\theta\leq\frac{\pi}{3}$$

2. Area integral.
   - Arrange and expand area integral:

$$A=4\int_\alpha^\beta \frac{1}{2}r(\theta)^2\,d\theta \quad \gg\gg \quad 4\int_{\pi/6}^{\pi/3}\frac{1}{2}\big(1+2\cos(4\theta)\big)^2\,d\theta$$

$$\gg\gg \quad 2\int_{\pi/6}^{\pi/3} 1+4\cos(4\theta)+4\cos^2(4\theta)\,d\theta$$

   - Simplify integral using power-to-frequency: $\cos^2 A \rightsquigarrow \frac{1}{2}(1+\cos(2A))$ with $A=4\theta$:

$$\gg\gg \quad 2\int_{\pi/6}^{\pi/3} 1+4\cos(4\theta)+4\cdot\frac{1}{2}\big(1+\cos(8\theta)\big)\,d\theta$$

   - Compute integral:

$$\gg\gg \quad 6\theta+2\sin(4\theta)+\frac{1}{4}\sin(8\theta)\Big|_{\pi/6}^{\pi/3}$$

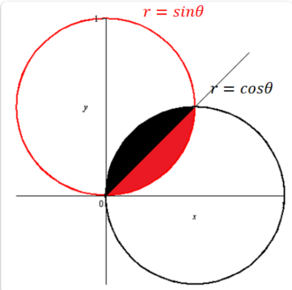
$$\gg\gg \quad \pi-\frac{3\sqrt{3}}{2}$$

Overlap area of circles

Compute the area of the overlap between crossing circles. For concreteness, suppose one of the circles is given by $r(\theta)=\sin\theta$ and the other is given by $r(\theta)=\cos\theta$.

Solution

Here is a drawing of the overlap:



1. Notice: total overlap area = 2× area of red region.
2. Bounds: $0 \leq \theta \leq \frac{\pi}{4}$.
3. Apply area formula for the red region.
   - Area formula applied to $r(\theta) = \sin\theta$:

$$A = \int_\alpha^\beta \frac{1}{2} r(\theta)^2 \, d\theta \qquad \gg\gg \qquad \int_0^{\pi/4} \frac{1}{2} \sin^2\theta \, d\theta$$

   - Power-to-frequency: $\sin^2\theta \rightsquigarrow \frac{1}{2}\big(1 - \cos(2\theta)\big)$:

$$\gg\gg \quad \int_0^{\pi/4} \frac{1}{4}\big(1 - \cos(2\theta)\big) \, d\theta$$

$$\gg\gg \quad \frac{1}{4}\theta - \frac{1}{8}\sin(2\theta)\Big|_0^{\pi/4} \quad \gg\gg \quad \frac{\pi}{16} - \frac{1}{8}$$

   - Double the result to include the black region:

$$\gg\gg \quad \frac{\pi}{8} - \frac{1}{4}$$