Name: Solutions

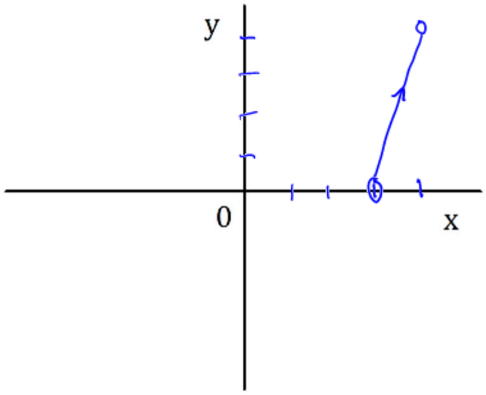
Worksheet 10.1 – Curves Defined by Parametric Equations

1) Express in the form $y = f(x)$, and sketch the graph of the parametric curve. (LT: 5a)

a) $x = t + 3, \quad y = 4t, \quad 0 < t < 1$

$t = x - 3$

$y = 4(x - 3)$




$y = 4x - 12$

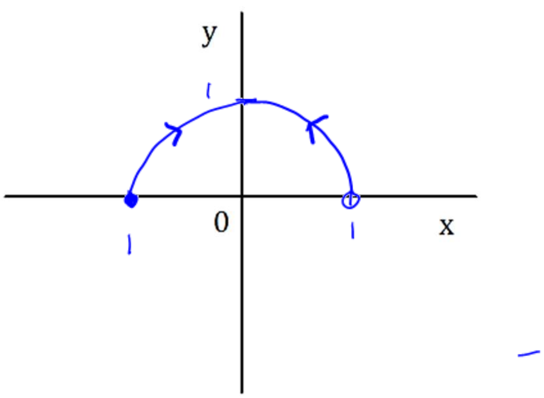
$3 < x < 4$

b) $x = \cos t, \quad y = \sin^2 t, \quad 0 < t < 2\pi$

$\cos^2 t + \sin^2 t = 1$

$x^2 + y = 1$

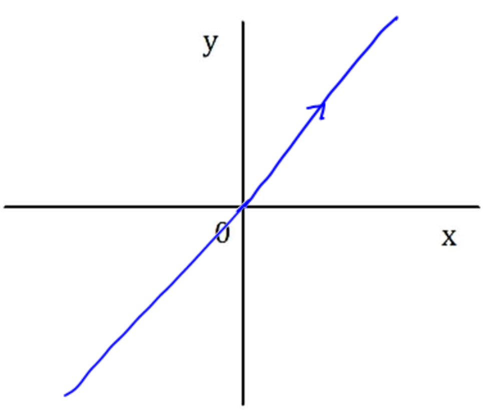
$y = 1 - x^2$




$y = y = 1 - x^2$

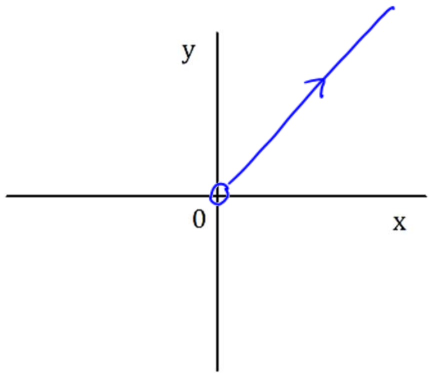
$-1 \leq x < 1$

c) x = t, y = t, $-\infty < t < \infty$




$y = x$

d) $x = e^t, y = e^t \quad -\infty < t < \infty$




$y = x$

$x > 0$

2) Find a parametrization c(t)= (x(t), y(t)) of the curve satisfying the given condition. (LT: 5b)

a) $y = 3x - 4, \quad c(0) = (2,2)$

There are many possible solutions

Here is one:

$x = t + 2$

$y = 3(t+2) - 4$

$= 3t + 2$

$c(t) = (\ t + 2 \quad , \ 3t + 2 \quad )$

b) $y = 3x - 4, \quad c(3) = (2,2)$

There are many possible solutions

$x = t - 1$

$y = 3(t-1) - 4$

$= 3t - 7$

$c(t) = (\ t - 1 \quad , \ 3t - 7 \quad )$

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Worksheet 10.2a – Calculus with Parametric Curves

1) Find the points on the curve $c(t) = (3t^2 - 2t, t^3 - 6t)$ where the tangent line has slope 3. (LT: 5f)

$\frac{dy}{dt} = 3t^2 - 6$

$\frac{dx}{dt} = 6t - 2$

Slope $= \frac{dy}{dx} = \frac{3t^2-6}{6t-2} = \frac{3}{2}\left(\frac{t^2-2}{3t-1}\right)$

$\frac{3}{2}\left(\frac{t^2-2}{3t-1}\right) = 3$

$t^2 - 2 = 2(3t-1)$

$t^2 - 6t = 0$

$t = 0, 6$

$t = 0 \to P(0,0)$

$t = 6 \to P(96, 180)$

Points: $(0,0)$ $(96,180)$

2) Find $\frac{d^2y}{dx^2}$ at t = 1 for $x = 4 - t^{-2}$ $y = t^{-1} + t$ (LT: 5f)

$\frac{dy}{dt} = -\frac{1}{t^2} + 1$

$\frac{dx}{dt} = 2t^{-3}$

$\frac{dy}{dx} = \frac{-\frac{1}{t^2}+1}{\frac{2}{t^3}} = -\frac{t}{2} + \frac{t^3}{2} = y'$

$\frac{d^2y}{dx^2} = \frac{\frac{d(y')}{dt}}{\frac{dx}{dt}} = \frac{-\frac{1}{2} + \frac{3}{2}t^2}{\frac{2}{t^3}} = -\frac{t^3}{4} + \frac{3}{4}t^5$

$\left.\frac{d^2y}{dx^2}\right|_{t=1} = -\frac{1}{4} + \frac{3}{4}$

$= \frac{1}{2}$

$\left.\frac{d^2y}{dx^2}\right|_{t=1} = \frac{1}{2}$

3) Find the t-interval(s) on which $c(t) = (t^2, t^3 - 4t)$ is concave up. (LT: 5f)

$\frac{dy}{dt} = 3t^2 - 4$

$\frac{dx}{dt} = 2t$

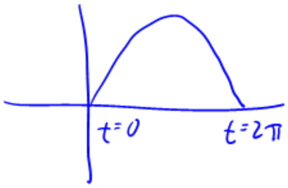
$\frac{dy}{dx} = y' = \frac{3t^2-4}{2t} = \frac{3}{2}t - \frac{2}{t}$

$\frac{d^2y}{dx^2} = \frac{\frac{d(y')}{dt}}{\frac{dx}{dt}} = \frac{\frac{3}{2} + \frac{2}{t^2}}{2t} = \frac{3}{4t} + \frac{1}{t^3} = \frac{1}{t}\left(\frac{3}{4} + \frac{1}{t^2}\right)$

$t > 0$

$\frac{1}{t}\left(\frac{3}{4} + \frac{1}{t^2}\right) > 0$ when $t > 0$

4) Let $c(t)=(x(t),y(t))$, where y(t) > 0 and $x'(t)>0$. Then the area under c(t) for $a \le t \le b$ is $A=\int_a^b y(t)x'(t)\,dt$. Find the area under one arch of the cycloid $c(t)=(5t-5\sin t, 5-5\cos t)$. (LT: 5g)




$\frac{dx}{dt} = 5-5\cos t$

$$A = \int_0^{2\pi} (5-5\cos t)(5-5\cos t)\,dt$$

$$= \int_0^{2\pi} (25 - 50\cos t + 25\cos^2 t)\,dt$$

$$= \int_0^{2\pi} \left[25 - 50\cos t + \frac{25}{2}(1+\cos 2t)\right]dt$$

$$= \int_0^{2\pi} \left(\frac{75}{2} - 50\overset{0}{\cancel{\cos t}} + \frac{25}{2}\overset{0}{\cancel{\cos 2t}}\right)dt$$

$$= \frac{75}{2}\Big|_0^{2\pi}$$

$$= 75\pi$$

$$= 25\pi$$

$$\boxed{75\pi}$$

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Worksheet 10.2b – Calculus with Parametric Curves

1) Find the length of the path, $x = 2t^2$, $y = 3t^2 - 1$ over $(0, 4)$. (LT: 5c)

$$\ell = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad \frac{dx}{dt} = 4t \qquad \frac{dy}{dt} = 6t$$

$$= \int_0^4 \sqrt{(4t)^2 + (6t)^2}\, dt$$

$$= \int_0^4 \sqrt{52t^2}\, dt$$

$$= \int_0^4 \sqrt{52}\, t\, dt$$

$$= \sqrt{52}\frac{t^2}{2}\Big|_0^4$$

$$= 8\sqrt{52} = 16\sqrt{13}$$

$$\boxed{16\sqrt{13}}$$

2) Find the minimum speed of a particle with trajectory $c(t) = (t^3 - 4t, t^2 + 1)$ for $t \geq 0$ where lengths are in cm and time is in seconds. Hint: It is easier to find the minimum of the square of the speed. (LT: 5e)

$$\text{Speed} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(3t^2-4)^2 + (2t)^2}$$

$$= \sqrt{9t^4 - 24t^2 + 16 + 4t^2}$$

$$= \sqrt{9t^4 - 20t^2 + 16}$$

$$(\text{Speed})^2 = 9t^4 - 20t^2 + 16$$

critical points: $\frac{d(\text{speed})^2}{dt} = 36t^3 - 40t = 0$

$$t(36t^2 - 40) = 0$$

$$t = 0 \qquad t^2 = \frac{40}{36} = \frac{10}{9}$$

$$t = \pm\frac{\sqrt{10}}{3}$$

Check $\quad \text{Speed}\Big|_{t=0} = \sqrt{16} = 4$

$$\text{speed}\Big|_{t=\frac{\sqrt{10}}{3}} = \sqrt{9\left(\frac{100}{81}\right) - 20\left(\frac{10}{9}\right) + 16}$$

$$= \sqrt{16 - \frac{100}{9}}$$

$$= \sqrt{\frac{144-100}{9}}$$

$$= \frac{\sqrt{44}}{3}$$

$$= \frac{2\sqrt{11}}{3} \quad \swarrow \text{minimum}$$

$$\boxed{\frac{2\sqrt{11}}{3} \text{ cm/s}}$$

3) Compute the length of one arch of the cycloid $c(t)=(t-\sin t, 1-\cos t)$. (LT: 5c)

$$\ell = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad \frac{dx}{dt} = 1-\cos t \qquad \frac{dy}{dt} = \sin t$$

$$= \int_0^{2\pi} \sqrt{(1-\cos t)^2 + \sin^2 t}$$

$$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$$

$$= \int_0^{2\pi} \sqrt{1-2\cos t+\cos^2 t+\sin^2 t}\, dt$$

$$1-\cos 2\theta = 2\sin^2\theta$$

$$1-\cos t = 2\sin^2\left(\frac{t}{2}\right)$$

$$= \int_0^{2\pi} \sqrt{2-2\cos t}\, dt$$

$$= \int_0^{2\pi} \sqrt{2\left(2\sin^2\left(\frac{t}{2}\right)\right)}\, dt$$

$$= \int_0^{2\pi} 2\sin\frac{t}{2}\, dt$$

$$= -4\cos\frac{t}{2}\Big|_0^{2\pi}$$

$$= 8$$

$$\boxed{8}$$

4) Compute the surface area generated by revolving one arch of the cycloid $c(t)=(t-\sin t, 1-\cos t)$ about the x-axis. (LT: 2e)

$$S = \int 2\pi r\, ds$$

$$= \int_0^{2\pi} 2\pi(1-\cos t)\left(2\sin\frac{t}{2}\right) dt$$

$$= \int_0^{2\pi} 2\pi\left(2\sin^2\frac{t}{2}\right)\left(2\sin\frac{t}{2}\right) dt$$

$$= \int_0^{2\pi} 8\pi \sin^3\frac{t}{2}\, dt$$

$$u = \frac{t}{2} \quad du = \frac{1}{2}dt$$

$$= \int_0^{\pi} 16\pi \sin^3 u\, du$$

$$= \int_0^{\pi} 16\pi(1-\cos^2 u)\sin u\, du$$

$$w = \cos u \quad dw = -\sin u\, du$$

$$= \int_1^{-1} -16\pi(1-w^2)\, dw$$

$$= 16\pi\left(w - \frac{w^3}{3}\right)\Big|_{-1}^{1}$$

$$= 16\pi\left[\left(1-\frac{1}{3}\right) - \left(-1+\frac{1}{3}\right)\right]$$

$$= 16\pi\left(\frac{4}{3}\right)$$

$$= \frac{64\pi}{3}$$

$$\boxed{\frac{64\pi}{3}}$$