W13 - Examples

Parametric curves

Parametric circles

The standard equation of a circle of radius R centered at the point (h, k):

$$(x-h)^2 + (y-k)^2 = R^2$$

This equation says that the *distance* from a point (x, y) on the circle to the center point (h, k) equals R. This fact defines the circle.

Parametric coordinates for the circle:

$$x = h + R\cos t, \qquad y = k + R\sin t, \qquad t \in [0, 2\pi)$$

For example, the unit circle $x^2 + y^2 = 1$ is parametrized by $x = \cos t$ and $y = \sin t$.

Parametric lines

Parametric coordinate functions for a line:

$$x = a + rt, \qquad y = b + st, \qquad t \in (-\infty, +\infty)$$

Compare this to the graph of linear function:

$$y = mx + b \qquad \text{some } m, b$$

Vertical lines cannot be described as the graph of a function. We must use $x = a$.

Parametric lines can describe all lines equally well, including horizontal and vertical lines.

A vertical line $x = a$ is achieved by setting $s = 0$ and $r \neq 0$.

A horizontal line $y = b$ is achieved by setting $r = 0$ and $s \neq 0$.

A non-vertical line $y = mx + b$ may be achieved by setting $s = m$ and $r = 1$, and $a = 0$.

Assuming that $r \neq 0$, the parametric coordinate functions describe a line satisfying:

$$y = b + s\left(\frac{x-a}{r}\right)$$

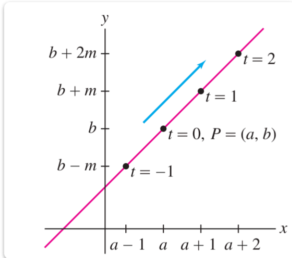
$$\gg\gg \quad y = \frac{s}{r} \cdot x + \left(b - \frac{s}{r} \cdot a\right)$$

and therefore the slope is $m = \frac{s}{r}$ and the y-intercept is $b - \frac{s}{r} \cdot a$.

The point-slope construction of a line has a parametric analogue:

point-slope line:

$$y - a = m(x - b) \qquad\qquad (x, y) = (a + t,\, b + mt)$$



Parametric ellipses

The general equation of an ellipse centered at (h, k) with half-axes a and b is:

$$\left(\frac{x-h}{a}\right)^2 + \left(\frac{y-k}{b}\right)^2 = 1$$

This equation represents a *stretched unit circle*:

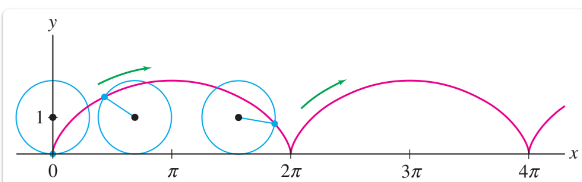
- by a in the x-axis
- by b in the y-axis

Parametric coordinate functions for the general ellipse:

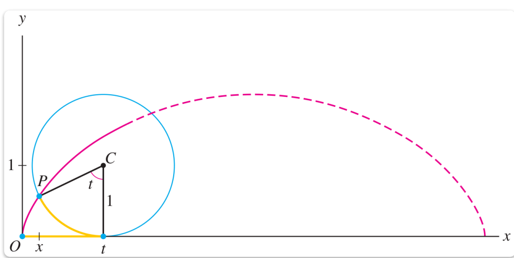
$$x = h + a\cos t, \qquad y = k + b\sin t, \qquad t \in [0, 2\pi)$$

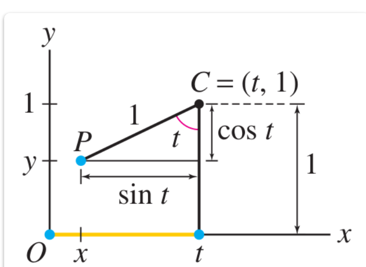
Parametric cycloids

The cycloid is the curve traced by a pen attached to the rim of a wheel as it rolls.



It is easy to describe the cycloid parametrically. Consider the geometry of the situation:





The center C of the wheel is moving rightwards at a constant speed of 1, so its position is $(t, 1)$. The angle is revolving at the same constant rate of 1 (in *radians*) because the *radius* is 1.

The triangle shown has base $\sin t$, so the x coordinate is $t - \sin t$. The y coordinate is $1 - \cos t$.

So the coordinates of the point $P = (x, y)$ are given parametrically by:

$$x = t - \sin t, \qquad y = 1 - \cos t, \qquad t > 0$$

If the circle has another radius, say R, then the parametric formulas change to:

$$x = Rt - R\sin t, \qquad y = R - R\cos t, \qquad t > 0$$

Calculus with parametric curves

Tangent to a cycloid

Find the tangent line (described parametrically) to the cycloid $(4t - 4\sin t,\, 4 - 4\cos t)$ when $t = \pi/4$.

Solution

Compute x' and y'.

Find $x'(t)$:

$$x(t) = 4t - 4\sin t \quad \gg\gg \quad x'(t) = 4 - 4\cos t$$

Find $y'(t)$:

$$y(t) = 4 - 4\cos t \quad \gg\gg \quad y'(t) = 4\sin t$$

Plug in $t = \pi/4$:

$$x'(\pi/4) \quad \gg\gg \quad 4 - 4\cos(\pi/4) \quad \gg\gg \quad 4 - 2\sqrt{2}$$

Plug in $t = \pi/4$:

$$y'(\pi/4) \quad \gg\gg \quad 4\sin(\pi/4) \quad \gg\gg \quad 2\sqrt{2}$$

Apply formula: $\dfrac{dy}{dx} = \dfrac{y'}{x'}$:

Calculate $\frac{dy}{dx}$ at $t = \pi/4$:

$$\frac{dy}{dx}(\pi/4) = \frac{y'(\pi/4)}{x'(\pi/4)} \qquad \gg\gg \qquad \frac{2\sqrt{2}}{4 - 2\sqrt{2}}$$

Simplify:

$$\gg\gg \qquad \frac{2\sqrt{2}}{4-2\sqrt{2}}\cdot\frac{4+2\sqrt{2}}{4+2\sqrt{2}}$$

$$\gg\gg \qquad \frac{8\sqrt{2}+8}{16-8} \qquad \gg\gg \qquad \sqrt{2}+1$$

So:

$$\left.\frac{dy}{dx}\right|_{t=\pi/4} = \sqrt{2}+1$$

This is the slope m for our line.

Need the point P for our line. Find (x, y) at $t = \pi/4$.

Plug $t = \pi/4$ into parametric formulas:

$$\big(x(t),\, y(t)\big)\Big|_{t=\pi/4} \quad \gg\gg \quad \Big(4\frac{\pi}{4} - 4\sin(\pi/4),\; 4-4\cos(\pi/4)\Big)$$

$$\gg\gg \quad \Big(\pi - 2\sqrt{2}, 4-2\sqrt{2}\Big)$$

Point-slope formulation of tangent line:

$$x = a + t, \quad y = b + mt$$

Inserting our data:

$$x \;=\; (\pi - 2\sqrt{2}) + t, \qquad y \;=\; (4-2\sqrt{2}) + (\sqrt{2}+1)t$$

Vertical and horizontal tangents of the circle

Consider the circle parametrized by $x = \cos t$ and $y = \sin t$. Find the points where the tangent lines are vertical or horizontal.

Solution

For the points with vertical tangent line, we find where the moving point has $x'(t) = 0$ (purely vertical motion):

$$x'(t) = -\sin t,$$

$$x'(t) = 0 \qquad \gg\gg \qquad -\sin t = 0 \qquad \gg\gg \qquad t = 0, \pi$$

The moving point is at $(1, 0)$ when $t = 0$, and at $(-1, 0)$ when $t = \pi$.

For the points with horizontal tangent line, we find where the moving point has $y'(t) = 0$ (purely horizontal motion):

$$y'(t) = \cos t,$$

$$y'(t) = 0 \qquad \gg\gg \qquad \cos t = 0$$

$$\gg\gg \qquad t = \frac{\pi}{2}, \frac{3\pi}{2}$$

The moving point is at $(0, 1)$ when $t = \pi/2$, and at $(0, -1)$ when $t = 3\pi/2$.

Finding the point with specified slope

Consider the parametric curve given by $(x, y) = (t^2, t^3)$. Find the point where the slope of the tangent line to this curve equals 5.

Solution

Compute the derivatives:

$$x'(t) = 2t, \qquad y'(t) = 3t^2$$

Therefore the slope of the tangent line, in terms of t:

$$m = \frac{dy}{dx} = \frac{y'(t)}{x'(t)}$$

$$\gg\gg \quad \frac{3t^2}{2t} \quad \gg\gg \quad \frac{3}{2}t$$

Set up equation:

$$m = 5$$

$$\frac{3}{2}t = 5$$

Solve. Obtain $t = \frac{10}{3}$.

Find the point:

$$(x, y)\Big|_{t=10/3} \quad \gg\gg \quad \left(\frac{100}{9}, \frac{1000}{27}\right)$$

Perimeter of a circle

The perimeter of the circle $(R\cos t, R\sin t)$ is easily found. We have $(x', y') = (-R\sin t, R\cos t)$, and therefore:

$$(x')^2 + (y')^2 = (-R\sin t)^2 + (R\cos t)^2$$

$$\gg\gg \qquad R^2\sin^2 t + R^2\cos^2 t \qquad \gg\gg \qquad R^2$$

$$ds = \sqrt{(x')^2 + (y')^2}\,dt = R\,dt$$

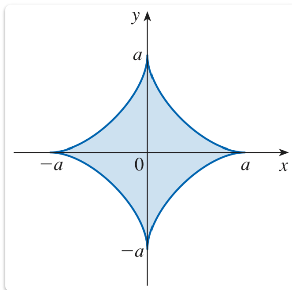
Integrate around the circle:

$$\text{Perimeter} \quad = \quad \int_0^{2\pi} ds \qquad \gg\gg \qquad \int_0^{2\pi} R\,dt$$

$$\gg\gg \qquad Rt\Big|_0^{2\pi} = 2\pi R$$

Perimeter of an asteroid

Find the perimeter length of the 'asteroid' given parametrically by $(x, y) = \big(a\cos^3\theta,\, a\sin^3\theta\big)$ for $a = 2$.



Solution

Notice: Throughout this problem we use the parameter θ instead of t. This does *not* mean we are using polar coordinates!

Compute the derivatives in θ:

$$(x', y') = \big(3a\cos^2\theta\sin\theta,\, 3a\sin^2\theta\cos\theta\big)$$

Compute the infinitesimal arc element.

$$(x')^2 + (y')^2 = 9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta$$

$$\gg\gg \qquad 9a^2\sin^2\theta\cos^2\theta\left(\cos^2\theta + \sin^2\theta\right)$$

$$\gg\gg \qquad 9a^2\sin^2\theta\cos^2\theta$$

Plug into the arc element, simplify:

$$ds = \sqrt{(x')^2 + y')^2}\,d\theta = \sqrt{9a^2\sin^2\theta\cos^2\theta}\,d\theta$$

$$\gg\gg \qquad ds = 3a\big|\sin\theta\cos\theta\big|\,d\theta$$

Bounds of integration?

Easiest to use $\theta \in [0, \pi/2]$. This covers one edge of the asteroid. Then multiply by 4 for the final answer.

On the interval $\theta \in [0, \pi/2]$, the factor $3a\sin\theta\cos\theta$ is *positive*. So we can drop the absolute value and integrate directly.

⚠ Absolute values matter!

If we tried to integrate on the whole range $\theta \in [0, 2\pi]$, then $3a \sin\theta \cos\theta$ really does change sign.

To perform integration properly with these absolute values, we'd need to convert to a piecewise function by adding appropriate minus signs.

Integrate the arc element:

$$\int_0^{\pi/2} ds \quad \gg\gg \quad \int_0^{\pi/2} 3a \sin\theta \cos\theta \, d\theta$$

$$\gg\gg \quad 3a \int_{u=0}^{1} u \, du \qquad (u = \sin\theta)$$

$$\gg\gg \quad 3a \frac{u^2}{2} \bigg|_0^1 \quad \gg\gg \quad \frac{3a}{2}$$

Finally, multiply by 4 to get the total perimeter: L=6a

Speed, distance, displacement

The parametric curve $\left(t, \frac{2}{3} t^{3/2}\right)$ describes the position of a moving particle (t measuring seconds).
(a) What is the speed function?

Suppose the particle travels for 8 seconds starting at $t = 0$.
(b) What is the total distance traveled?
(c) What is the total displacement?

Solution

(a)
Compute *derivatives*:

$$(x', y') = \left(1, t^{1/2}\right)$$

Compute the *speed*.

Find sum of squares:

$$(x')^2 + (y')^2 = 1 + (t^{1/2})^2 = 1 + t$$

Get the speed function:

$$v(t) = \sqrt{(x')^2 + (y')^2} = \sqrt{1+t}$$

(b)

Distance traveled by using *speed*.

Compute total distance traveled function:

$$s(t) = \int_{u=0}^{t} \sqrt{1+u}\, du$$

Integrate.

Substitute $w = 1+u$ and $dw = du$.

New bounds are 1 and $1+t$.

Calculate:

$$\gg\gg \quad \int_1^{1+t} \sqrt{w}\, dw$$

$$\gg\gg \quad \frac{2}{3} w^{3/2} \Big|_1^{1+t} \quad \gg\gg \quad \frac{2}{3}\left((1+t)^{3/2} - 1\right)$$

Insert $t = 8$ for the answer.

The distance traveled up to $t = 8$ is:

$$s(8) = \frac{2}{3}\left(9^{3/2} - 1\right) \quad \gg\gg \quad \frac{2}{3}(27-1) \quad \gg\gg \quad \frac{52}{3}$$

This is our final answer.

(c)

Displacement formula: $d = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$

Pythagorean formula for distance between given points.

Compute starting and ending points.

For starting point, insert $t = 0$:

$$\big(x(t), y(t)\big)\Big|_{t=0} \quad \gg\gg \quad \left(t, \frac{2}{3} t^{3/2}\right)\Big|_{t=0} \quad \gg\gg \quad (0,0)$$

For ending point, insert $t = 8$:

$$\big(x(t), y(t)\big)\Big|_{t=8} \quad \gg\gg \quad \left(t, \frac{2}{3} t^{3/2}\right)\Big|_{t=8}$$

$$\gg\gg \quad \left(8, \frac{2}{3} 8^{3/2}\right) \quad \gg\gg \quad \left(8, \frac{32\sqrt{2}}{3}\right)$$

Plug points into distance formula.

Insert $(0,0)$ and $\left(8, 32\sqrt{2}/3\right)$:

$$\sqrt{8^2 + \left(\frac{32\sqrt{2}}{3}\right)^2} \quad \gg\gg \quad \sqrt{64 + \frac{2048}{9}}$$

$$\gg\gg \quad \frac{\sqrt{2624}}{3}$$

This is our final answer.

Surface of revolution - parametric circle

By revolving the unit upper semicircle about the x-axis, we can compute the surface area of the unit sphere.

The parametrization of the unit upper semicircle is: $(x,y) = (\cos t,\, \sin t)$.

The derivative is: $(x',y') = (-\sin t,\, \cos t)$.

Therefore, the arc element:

$$ds = \sqrt{(x')^2 + (y')^2}\, dt$$

$$\gg\gg \quad \sqrt{(-\sin t)^2 + (\cos t)^2}\, dt \quad \gg\gg \quad dt$$

Now for R we choose $R = y(t) = \sin t$ because we are revolving about the x-axis.

Plugging all this into the integral formula and evaluating gives:

$$A = \int_0^\pi 2\pi \sin t\, dt \quad \gg\gg \quad -2\pi \cos t\Big|_0^\pi \quad \gg\gg \quad 4\pi$$

Notice: This method is a little easier than the method using the graph $y = \sqrt{1-x^2}$.

Surface of revolution - parametric curve

Set up the integral which computes the surface area of the surface generated by revolving about the x-axis the curve $(t^3,\, t^2 - 1)$ for $0 \le t \le 1$.

Solution

For revolution about the x-axis, we set $R = y(t) = t^2 - 1$.

Then compute ds:

$$ds = \sqrt{(x')^2 + (y')^2} \quad \gg\gg \quad \sqrt{(3t^2)^2 + (2t)^2} \quad \gg\gg \quad \sqrt{9t^4 + 4t^2}$$

$$\gg\gg \quad \sqrt{t^2(9t^2 + 4)} \quad \gg\gg \quad t\sqrt{9t^2 + 4}$$

Therefore the desired integral is:

$$A = \int_0^1 2\pi R\, ds \quad \gg\gg \quad \int_0^1 2\pi(t^2-1)t\sqrt{9t^2+4}\, dt$$