Name: Solutions

Worksheet 11.9 – Representations of Functions as Power Series

1) (LT: 4e) Use $\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$ for $|x|<1$ to expand the function in a power series with center $a=0$ and determine the interval of convergence.

a) $f(x)=\frac{1}{5-x}$ $=\frac{1}{5(1-\frac{x}{5})}=\frac{1}{5}\left(\frac{1}{1-\frac{x}{5}}\right)$

$T(x)=\frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n$

Convergent for $\left|\frac{x}{5}\right|<1$
$|x|<5$

T(x) = $\sum_{n=0}^{\infty}\frac{1}{5^{n+1}}x^n$

Interval: $(-5,5)$

b) $f(x)=\frac{1}{16+2x^3}$ $=\frac{1}{16(1+\frac{2x^3}{16})}=\frac{1}{16}\left(\frac{1}{1-(-\frac{x^3}{8})}\right)$

$T(x)=\frac{1}{16}\sum_{n=0}^{\infty}\left(\frac{-x^3}{8}\right)^n=\sum_{n=0}^{\infty}\frac{(-1)^n}{16\cdot 8^n}x^{3n}$

$=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{3n+4}}x^{3n}$

Convergent for $\left|-\frac{x^3}{8}\right|<1 \longrightarrow |x|<2$

T(x) = $\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{3n+4}}x^{3n}$

Interval: $(-2,2)$

2) (LT: 4g) If the power series for a function, $f(x)$, is $\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+2}}{n+1}$ with R = 1, what is the power series for $f'(x)$ and what is the radius of convergence?

$T(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}(2n+2)x^{2n+1}$

$=\sum_{n=0}^{\infty}(-1)^n(2)x^{2n+1}$

$R=1$

T(x) = $\sum_{n=0}^{\infty}(-1)^n(2)x^{2n+1}$

R = 1

3) (LT: 4e, h) Find the power series representation for a) $f(x)=\frac{1}{1+x^4}$ and use it to find a power series representation for b) $\int f(x)\,dx$.

$\frac{1}{1+x^4}=\frac{1}{1-(-x^4)}$ $T(x)=\sum_{n=0}^{\infty}(-x^4)^n$

$\int\frac{1}{1+x^4}dx=C+\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+1}}{4n+1}$

a) T(X) = $\sum_{n=0}^{\infty}(-1)^n x^{4n}$

b) T(X) = $C+\sum_{n=0}^{\infty}\frac{(-1)^n}{4n+1}x^{4n+1}$

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Worksheet 11.10a – Taylor and Maclaurin Series

1) (LT: 4d) Write out the first four terms of the Taylor series of *f(x)* centered at a = 3 if

$f(3)=1 \quad f'(3)=2 \quad f''(3)=12 \quad f'''(3)=3$

$c_0 = f(3) = 1$ $\quad c_2 = \frac{f''(3)}{2!} = 6$

$c_1 = f'(3) = 2$ $\quad c_3 = \frac{f'''(3)}{3!} = \frac{3}{6} = \frac{1}{2}$

$$\boxed{1 + 2(x-3) + 6(x-3)^2 + \frac{1}{2}(x-3)^3 + \cdots}$$

2) (LT: 4d) Find the Taylor series centered at a = 1 for the function, $f(x) = \frac{1}{x}$. Also find the interval on which the expansion is valid.

n	$f^{(n)}(x)$	$f^{(n)}(1)$	c_n
0	$\frac{1}{x}$	1	1
1	$-\frac{1}{x^2}$	-1	-1
2	$\frac{2}{x^3}$	2	1
3	$\frac{-6}{x^4}$	-6	-1

⋮

$c_n = (-1)^n$

$T(x) = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$

geometric

$|r| = |x-1|$

convergent for $|x-1|<1$

divergent otherwise

T(x) = $\sum_{n=0}^{\infty} (-1)^n (x-1)^n$

Interval: $(0, 2)$

3) (LT: 4e) Find the Maclaurin series and find the interval on which the expansion is valid.

a) $f(x) = \sin(3x^2)$

$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$T(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (3x^2)^{2n+1}}{(2n+1)!}$

T(x) = $\sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1}}{(2n+1)!} x^{4n+2}$

Interval: $(-\infty, \infty)$

b) $f(x) = x^2 e^{5x}$

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$T(x) = x^2 \sum_{n=0}^{\infty} \frac{(5x)^n}{n!}$

T(x) = $\sum_{n=0}^{\infty} \frac{5^n}{n!} x^{n+2}$

Interval: $(-\infty, \infty)$

c) $f(x) = x\ln(1-5x)$

$\ln(1-x) = \sum_{n=0}^{\infty} -\frac{x^{n+1}}{n+1}$

$[-1, 1)$

$T(x) = x \sum_{n=0}^{\infty} \frac{-(5x)^{n+1}}{n+1}$

Convergent for $-1 \le 5x < 1$

$-\frac{1}{5} \le x < \frac{1}{5}$

T(x) = $\sum_{n=0}^{\infty} -\frac{5^{n+1}}{n+1} x^{n+2}$

Interval: $\left[-\frac{1}{5}, \frac{1}{5}\right)$

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Worksheet 11.10b – Taylor and Maclaurin Series

1) (LT: 4e) Identify the function, $f(x)$, for each of these Maclaurin series.

a) $\sum_{n=0}^{\infty}(-1)^n 2^n x^n$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$$\sum_{n=0}^{\infty} (-1)^n 2^n x^n = \sum_{n=0}^{\infty} (-2x)^n = \frac{1}{1-(-2x)}$$

$f(x) = \dfrac{1}{1+2x}$

b) $\sum_{n=0}^{\infty}(-1)^n \frac{x^{3n}}{n!}$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{3n}}{n!} = \sum_{n=0}^{\infty} \frac{(-x^3)^n}{n!} = e^{-x^3}$$

$f(x) = e^{-x^3}$

c) $\sum_{n=0}^{\infty}(-1)^n \frac{5x^{4n+2}}{(2n+1)!}$

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{5x^{4n+2}}{(2n+1)!} = 5\sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!}$$

$$= 5\sin(x^2)$$

$f(x) = 5\sin(x^2)$

d) $\sum_{n=0}^{\infty} \frac{(-5x)^{n+1}}{n+1}$

$$\sum_{n=0}^{\infty} -\frac{x^{n+1}}{n+1} = \ln(1-x)$$

$$\sum_{n=0}^{\infty} \frac{(-5x)^{n+1}}{n+1} = -\sum_{n=0}^{\infty} -\frac{(-5x)^{n+1}}{n+1} = -\ln(1-(-5x))$$

$$= -\ln(1+5x)$$

$f(x) = -\ln(1+5x)$

2) (LT: 4e) Find the sum, s, of these series.

a) $\sum_{n=0}^{\infty} \frac{(-5)^n}{n!}$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

$$\sum_{n=0}^{\infty} \frac{(-5)^n}{n!} = e^{-5}$$

s = e^{-5}

b) $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{9^n (2n)!}$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \cos x$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{9^n (2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{3^{2n}(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{3}\right)^{2n}}{(2n)!}$$

$$= \cos \frac{\pi}{3} = \frac{1}{2}$$

s = $\frac{1}{2}$

3) (LT: 4d, e) Determine the Maclaurin series for the function, $f(x) = x^2 \sin(5x^3)$, and use it to find $f^{(35)}(0)$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$x^2 \sin(5x^3) = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n (5x^3)^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n+1} x^{6n+5}}{(2n+1)!}$$

$$C_{35} = \frac{(-1)^5 5^{2(5)+1}}{(2(5)+1)!} \quad (n=5)$$

$$= \frac{-1(5^{11})}{11!}$$

$$\frac{f^{(35)}(0)}{35!} = \frac{-5^{11}}{11!}$$

$$f^{(35)}(0) = \frac{-5^{11}(35!)}{11!}$$

T(x) = $\sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n+1}}{(2n+1)!} x^{6n+5}$

$f^{(35)}(0) = \frac{-5^{11}(35!)}{11!}$

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Worksheet 11.11 – Applications of Taylor Polynomials

1) (LT: 4f) Estimate $\cos(0.02)$ with an error less than $1\text{x}10^{-6}$ and with a minimum number of terms. (Without a calculator!)

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

$$\cos(0.02) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (0.02)^n = 1 - \frac{(0.02)^2}{2} + \frac{(0.02)^4}{4!} - \frac{(0.02)^6}{6!} - \cdots$$

$$\cos(0.02) \approx 1 - \frac{(0.02)^2}{2} \qquad |\text{error}| < \frac{(0.02)^4}{4!} < 1\times 10^{-6}$$

$$\approx 1 - 0.0002$$

$$\boxed{0.9998}$$

2) (LT: 4e – f)Use the power series for $f(x) = e^x$ to show that $\frac{1}{e} = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots$, and then approximate e^{-1} to within an error of at most 10^{-3}.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots$$

$$= \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots$$

$$= \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} - \frac{1}{7(720)} + \cdots$$

$$e^{-1} \approx \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} \quad \text{with } |\text{error}| < \frac{1}{7(720)} < 10^{-3}$$

$$e^{-1} \approx \frac{360 - 120 + 30 - 6 + 1}{720}$$

$$e^{-1} \approx \frac{265}{720} = \frac{53}{144}$$

$$\begin{array}{r} .3680\overline{5} \\ 144\,)\,\overline{53.0000} \\ 432 \\ \hline 980 \\ 864 \\ \hline 1160 \\ 1152 \\ \hline 800 \end{array}$$

$$\boxed{0.3681}$$

3) (LT: 4e, f, g) Express $\int_0^1 sin(x^2)dx$ a) as an infinite series and b) estimate its value to within an error of at most 10^{-3}.

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n \, x^{2n+1}}{(2n+1)!}$$

$$\sin(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{4n+2}$$

$$\int_0^1 \sin(x^2)dx = \int_0^1 \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{4n+2}\right] dx$$

$$= \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{x^{4n+3}}{4n+3}\right]_0^1$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{1}{4n+3}\right)$$

$$= \frac{1}{3} - \frac{1}{3!(7)} + \frac{1}{5!(11)} - \cdots$$

$$\approx \frac{1}{3} - \frac{1}{42} \qquad |\text{error}| < \frac{1}{5!(11)} < 1\times 10^{-3}$$

$$\frac{1}{3} - \frac{1}{42} = \frac{13}{42}$$

$$\begin{array}{r} 0.30952 \\ 42 \overline{)\,13.0000} \\ 12\,6 \\ \hline 400 \\ 378 \\ \hline 220 \\ 210 \\ \hline 100 \\ 84 \\ \hline 16 \end{array}$$

a) $\int_0^1 sin(x^2)dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{1}{4n+3}\right)$

b) $\int_0^1 sin(x^2)dx \approx 0.3095$