W10 - Examples

Ratio test and Root test

Ratio test examples

(a) Observe that $\sum_{n=0}^{\infty} \frac{10^n}{n!}$ has ratio $R_n = \frac{10}{n+1}$ and thus $R_n \to 0 = L < 1$. Therefore the RaT implies that this series converges.

> ⚠ **Notice this technique!**
>
> Simplify the ratio:
>
> $$\frac{\frac{10^{n+1}}{(n+1)!}}{\frac{n!}{10^n}} \quad \gg\gg \quad \frac{(n+1)!}{10^{n+1}} \cdot \frac{n!}{10^n}$$
>
> $$\gg\gg \quad \frac{10 \cdot 10^n}{(n+1)n!} \cdot \frac{n!}{10^n} \quad \gg\gg \quad \frac{10}{n+1} \quad \xrightarrow{n\to\infty} \quad 0$$
>
> We *frequently* use these rules:
>
> $$10^{n+1} = 10^n \cdot 10, \qquad (n+1)! = (n+1)n!$$
>
> to simplify ratios having exponents and factorials.

(b) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ has ratio $R_n = \frac{(n+1)^2}{2^{n+1}} \Big/ \frac{n^2}{2^n}$.

Simplify this:

$$\frac{(n+1)^2}{2^{n+1}} \Big/ \frac{n^2}{2^n} \qquad \gg\gg \qquad \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2}$$

$$\gg\gg \qquad \frac{(n+1)^2 \cdot 2^n}{n^2 \cdot 2 \cdot 2^n} \qquad \gg\gg \qquad \frac{n^2+2n+1}{2n^2} \xrightarrow{n\to\infty} \frac{1}{2} = L$$

So the series *converges absolutely* by the ratio test.

(c) Observe that $\sum_{n=1}^{\infty} n^2$ has ratio $R_n = \dfrac{n^2+2n+1}{n^2} \to 1$ as $n \to \infty$.

So the ratio test is *inconclusive*, even though this series fails the SDT and obviously diverges.

(d) Observe that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ has ratio $R_n = \dfrac{n^2}{n^2+2n+1} \to 1$ as $n \to \infty$.

So the ratio test is *inconclusive*, even though the series converges as a p-series with $p = 2 > 1$.

(e) More generally, the ratio test is usually *inconclusive for rational functions*; it is more effective to use LCT with a p-series.

Root test examples

(a) Observe that $\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^n$ has roots of terms:

$$|a_n|^{1/n} = \left(\left(\frac{1}{n}\right)^n\right)^{1/n} = \frac{1}{n} \overset{n\to\infty}{\longrightarrow} 0 = L$$

Because $L < 1$, the RooT shows that the series converges absolutely.

(b) Observe that $\sum_{n=1}^{\infty} (-1)^n \left(\frac{n}{2n+1}\right)^n$ has roots of terms:

$$\sqrt[n]{|a_n|} = \frac{n}{2n+1} \overset{n\to\infty}{\longrightarrow} \frac{1}{2} = L$$

Because $L < 1$, the RooT shows that the series converges absolutely.

(c) Observe that $\sum_{n=1}^{\infty} \left(\frac{3}{n}\right)^n$ converges because $\sqrt[n]{|a_n|} = \frac{3}{n} \to 0$ as $n \to \infty$.

Ratio test versus root test

Determine whether the series $\sum_{n=1}^{\infty} \frac{n^2 4^n}{5^{n+2}}$ converges absolutely or conditionally or diverges.

Solution

Before proceeding, rewrite somewhat the general term as $\left(\frac{n}{5}\right)^2 \cdot \left(\frac{4}{5}\right)^n$.

Now we solve the problem first using the ratio test. By plugging in $n+1$ we see that

$$a_{n+1} = \left(\frac{n+1}{5}\right)^2 \cdot \left(\frac{4}{5}\right)^{n+1}$$

So for the ratio R_n we have:

$$\left(\frac{n+1}{5}\right)^2 \cdot \left(\frac{4}{5}\right)^{n+1} \cdot \left(\frac{5}{n}\right)^2 \cdot \left(\frac{5}{4}\right)^n$$

$$\gg\gg \qquad \frac{n^2+2n+1}{n^2} \cdot \frac{4}{5} \longrightarrow \frac{4}{5} < 1 \text{ as } n \to \infty$$

Therefore the series converges absolutely by the ratio test.

Now solve the problem again using the root test. We have for $\sqrt[n]{|a_n|}$:

$$\left(\left(\frac{n}{5}\right)^2 \cdot \left(\frac{4}{5}\right)^n\right)^{1/n} = \left(\frac{n}{5}\right)^{2/n} \cdot \frac{4}{5}$$

To compute the limit as $n \to \infty$ we must use logarithmic limits and L'Hopital's Rule. So, first take the log:

$$\ln\left(\left(\frac{n}{5}\right)^{2/n} \cdot \frac{4}{5}\right) = \frac{2}{n} \ln\frac{n}{5} + \ln\frac{4}{5}$$

Then for the first term apply L'Hopital's Rule:

$$\frac{\ln\frac{n}{5} \overset{d/dx}{\longrightarrow} \frac{1}{n/5} \cdot \frac{1}{5}}{n/2 \overset{d/dx}{\longrightarrow} 1/2} \qquad \gg\gg \qquad \frac{1/n}{1/2} \qquad \gg\gg \qquad \frac{2}{n} \quad \longrightarrow 0 \text{ as } n \to \infty$$

So the first term goes to zero, and the second (constant) term is the value of the limit. So the log limit is $\ln\frac{4}{5}$, and the limit (before taking logs) must be $e^{\ln\frac{4}{5}}$ (inverting the log using e^x) and this is $\frac{4}{5}$. Since $\frac{4}{5} < 1$, the root test also shows that the series converges absolutely.

Power series: Radius and Interval

Radius of convergence

Find the radius of convergence of the series:

(a) $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$ (b) $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

Solution

(a) The ratio of coefficients is $R_n = \left|\frac{a_{n+1}}{a_n}\right| = \frac{1/2^{n+1}}{1/2^n} = 1/2$.

Therefore $R = 2$ and the series converges for $|x| < 2$.

(b) This power series has $a_{2n+1} = 0$, meaning it skips all odd terms.

Instead of the standard ratio function, we take the ratio of successive *even terms*. The series of even terms has coefficients $a_n = \frac{1}{(2n)!}$. So:

$$\left|\frac{a_{n+1}}{a_n}\right| \quad \gg\gg \quad \frac{\frac{1}{(2(n+1))!}}{\frac{1}{(2n)!}}$$

$$\gg\gg \quad \frac{1}{(2n+2)(2n+1)(2n)!} \cdot \frac{(2n)!}{1} \quad \gg\gg \quad \frac{1}{(2n+2)(2n+1)}$$

As $n \to \infty$, this converges to 0, so $L = 0$ and $R = \infty$.

Radius and interval for a few series

Series	Radius	Interval
$\sum_{n=0}^{\infty} x^n$	$R = 1$	$(-1, 1)$
$\sum_{n=1}^{\infty} \frac{(x-2)^n}{n}$	$R = 1$	$[1, 3)$
$\sum_{n=0}^{\infty} n!\, x^n$	$R = 0$	$\{0\}$
$\sum_{n=0}^{\infty} \frac{x^n}{(2n)!}$	$R = \infty$	$(-\infty, \infty)$

Interval of convergence

Find the interval of convergence of the following series.

(a) $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ (b) $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$

Solution

(a) $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$

1. Apply ratio test.
   - Ratio of successive coefficients:

$$R_n = \left|\frac{1}{n+1} \cdot \frac{n}{1}\right| \quad \gg\gg \quad \frac{n}{n+1}$$

- Limit of ratios:

$$R_n = \frac{n}{n+1} \overset{n\to\infty}{\longrightarrow} 1$$

- Deduce $L = 1$ and therefore $R = 1$.
- Therefore:

$$|x-3| < 1 \implies \text{converges}$$

$$|x-3| > 1 \implies \text{diverges}$$

2. Preliminary interval of convergence.
   - Translate to interval notation:

$$|x-3| < 1 \quad \gg\gg \quad x \in (3-1, 3+1)$$

$$\gg\gg \quad x \in (2, 4)$$

3. Final interval of convergence.
   - Check endpoint $x = 2$:

$$\sum_{n=1}^{\infty} \frac{(2-3)^n}{n} \quad \gg\gg \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

$$\gg\gg \quad \text{converges by AST}$$

   - Check endpoint $x = 4$:

$$\sum_{n=1}^{\infty} \frac{(4-3)^n}{n} \quad \gg\gg \quad \sum_{n=1}^{\infty} \frac{1}{n}$$

$$\gg\gg \quad \text{diverges as } p\text{-series}$$

   - Final interval of convergence: $x \in [2, 4)$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$

1. Limit of coefficients ratio.
   - Ratio of successive coefficients:

$$R_n = \left|\frac{a_{n+1}}{a_n}\right| \quad \gg\gg \quad \left|\frac{(-3)^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-3)^n}\right|$$

$$\gg\gg \quad \frac{3\sqrt{n+1}}{\sqrt{n+2}}$$

   - Limit of ratios:

$$\lim_{n\to\infty} R_n \quad \gg\gg \quad \lim_{n\to\infty} \frac{3\sqrt{n+1}}{\sqrt{n+2}} \quad \gg\gg \quad 3$$

   - Deduce $L = 3$ and thus $R = 1/3$.
   - Therefore:

$$|x| < \frac{1}{3} \implies \text{converges}$$

$$|x| > \frac{1}{3} \implies \text{diverges}$$

   - Preliminary interval of convergence: $x \in \left(-\frac{1}{3}, \frac{1}{3}\right)$

2. Check endpoints.
   - Check endpoint $x = -1/3$:

$$\sum_{n=0}^{\infty} \frac{\left(-3\cdot\left(-\frac{1}{3}\right)\right)^n}{\sqrt{n+1}} \quad \gg\gg \quad \sum_{n=0}^{\infty} \frac{1^n}{\sqrt{n+1}}$$

$$\gg\gg \quad \text{diverges by LCT with } b_n = 1/\sqrt{n}$$

   - Check endpoint $x = +1/3$:

$$\sum_{n=0}^{\infty} \frac{\left(-3\cdot\left(+\frac{1}{3}\right)\right)^n}{\sqrt{n+1}} \quad \gg\gg \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$$

$$\gg\gg \quad \text{converges by AST}$$

   - Final interval of convergence: $x \in (-1/3, 1/3]$

Interval of convergence - further examples

Find the interval of convergence of the following series.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(4x+1)^n}{n}$

Solution

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$

- Ratio of coefficients: $R_n = \dfrac{n+1}{3n} \longrightarrow \dfrac{1}{3}$.
- So the $R = 3$, center is $x = -2$, and the preliminary interval is $(-2-3, -2+3) = (-5, 1)$.
- Check endpoints: $\sum \dfrac{n(-3)^n}{3^{n+1}}$ diverges and $\sum \dfrac{n(3)^n}{3^{n+1}}$ also diverges. Final interval is $(-5, 1)$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(4x+1)^n}{n}$

- Ratio of coefficients: $R_n = \dfrac{n+1}{n} \longrightarrow 1$.
- So $R = 1$, and the series converges when $|4x+1| < 1$.
- Extract preliminary interval.
  - Divide by 4:

$$|4x+1| < 1 \quad \overset{\div 4}{\gg\gg} \quad |x + 1/4| < 1/4 \quad \gg\gg \quad x \in (0, 1/2)$$

- Check endpoints: $\sum \dfrac{(4\cdot\frac{-1}{2}+1)^n}{n}$ converges but $\sum \dfrac{1}{n}$ diverges.
- Final interval of convergence: $[-1/2, 0)$