Name: Solutions

Worksheet 11.3 – The Integral Test and Estimates of Sums

1) Formally show whether the infinite series is convergent. (LT: 4b)

a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2}}$ is a divergent p-series

$p = \frac{2}{3} \leq 1$ (or $p = \frac{2}{3} \not> 1$)

b) $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

$f(x) = \frac{1}{x^2+1}$ is continuous, positive and decreasing on $[1, \infty)$

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{R \to \infty} \int_1^R \frac{1}{x^2+1}\,dx$$

$$= \lim_{R \to \infty} \tan^{-1} x \Big|_1^R$$

$$= \lim_{R \to \infty} \left[\tan^{-1} R - \tan^{-1} 1\right]$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

$\int_1^{\infty} \frac{1}{x^2+1}\,dx$ is convergent

so $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ is convergent by Integral Test

c) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

$f(x) = \frac{1}{x(\ln x)^2}$ is continuous, positive, and decreasing on $[2, \infty)$

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx = \lim_{R \to \infty} \int_2^R \frac{1}{x(\ln x)^2}\,dx$$

$u = \ln x \quad du = \frac{1}{x}\,dx$

$$= \lim_{R \to \infty} \int_{\ln 2}^{\ln R} u^{-2}\,du$$

$$= \lim_{R \to \infty} \left[-\frac{1}{u}\Big|_{\ln 2}^{\ln R}\right]$$

$$= \lim_{R \to \infty} \left[-\frac{1}{\ln R} + \frac{1}{\ln 2}\right]$$

$$= \frac{1}{\ln 2}$$

$\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx$ is convergent

so $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ is convergent by Integral Test

Name: Solutions

Worksheet 11.4 – The Comparison Tests

1) Use the Comparison Test to show whether the series is convergent or divergent. (LT: 4b)

a) $\sum_{n=1}^{\infty} \frac{1}{n^{1/3}+2^n}$

$a_n = \frac{1}{n^{1/3}+2^n}$ $\quad b_n = \frac{1}{2^n}$

$0 \le a_n \le b_n$

$\sum b_n$ is a convergent geometric series, $|r| = \frac{1}{2} < 1$

$\sum a_n$ is convergent by CT

b) $\sum_{k=2}^{\infty} \frac{\sqrt{k}}{k-1}$

$a_K = \frac{\sqrt{K}}{K-1}$ $\quad b_K = \frac{\sqrt{K}}{K} = \frac{1}{\sqrt{K}}$ $\quad$ (can also use $b_n = \frac{1}{K}$)
$\downarrow p = 1$

$0 \le b_K \le a_K$

$\sum b_K$ is a divergent p-series, $p = \frac{1}{2} \not> 1$

$\sum_{K=2}^{\infty} a_K$ is divergent by CT

2) Use the Limit Comparison Test to show whether the series is convergent or divergent. (LT:4b)

a) $\sum_{n=2}^{\infty} \frac{n^2}{n^4-1}$

$a_n = \frac{n^2}{n^4-1} \ge 0$ $\quad b_n = \frac{n^2}{n^4} = \frac{1}{n^2} \ge 0$

$\sum b_n$ is a convergent p-series, $p = 2 > 1$

$\lim_{n\to\infty} \frac{b_n}{a_n} = \lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{n^2}{n^4-1}} = \lim_{n\to\infty} \frac{n^4-1}{n^4} = 1 \neq 0$; finite

$\sum_{n=2}^{\infty} a_n$ is convergent by LCT

b) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\ln n}$

$a_n = \frac{1}{\sqrt{n}+\ln n} \ge 0$ $\quad b_n = \frac{1}{\sqrt{n}} \ge 0$

$\sum b_n$ is a divergent p-series, $p = \frac{1}{2} \not> 1$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}+\ln n}}{\left(\frac{1}{\sqrt{n}}\right)} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}+\ln n} = 1 \neq 0$; finite

$\sum a_n$ is divergent by LCT

Name: Solutions

Worksheet 11.5 – Alternating Series and Absolute Convergence

1) Show whether the following series are absolutely convergent, conditionally convergent, or divergent. (LT: 4b)

a) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{1/3}}$

$a_n = \frac{(-1)^{n-1}}{n^{1/3}}$ $b_n = |a_n| = \frac{1}{n^{1/3}}$

$\sum b_n$ is a divergent p-series, $p = \frac{1}{3} \not> 1$ so $\sum a_n$ is NOT AC

$b_{n+1} \le b_n$ $\left(\frac{1}{(n+1)^{1/3}} \le \frac{1}{n^{1/3}}\right)$

$\lim_{n\to\infty} b_n = 0$ $\left(\lim_{n\to\infty} \frac{1}{n^{1/3}} = 0\right)$ $\rightarrow$ $\sum a_n$ is convergent by AST

$\boxed{CC}$

b) $\sum_{n=1}^{\infty} \frac{(-1)^n n^4}{n^3+1}$

$a_n = \frac{(-1)^n n^4}{n^3+1}$

$\lim_{n\to\infty} a_n \neq 0$ (DNE)

$\sum a_n$ is $\boxed{\text{divergent}}$ by DT

c) $\sum_{n=1}^{\infty} \frac{\cos \pi n}{n^3+1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3+1}$

$a_n = \frac{(-1)^n}{n^3+1}$ $b_n = |a_n| = \frac{1}{n^3+1}$ $c_n = \frac{1}{n^3}$

$\sum c_n$ is a convergent p-series, $p = 3 > 1$

$0 \le b_n \le c_n$

$\sum b_n$ is convergent by CT

so $\sum a_n$ is $\boxed{AC}$

2) Approximate $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!}$ such that $|error| < 0.005$. (LT: 4f)

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} = 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \frac{1}{120} - \frac{1}{720} + \cdots$$

$$|error| < \frac{1}{720} < \frac{1}{200}$$

$\frac{1}{200} = 0.005$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} \approx 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \frac{1}{120}$$

$$\approx \frac{120 - 60 + 20 - 5 + 1}{120} = \frac{76}{120} = \frac{19}{30}$$

$$30 \overline{)19.0000} = 0.63333$$

180000

100

90

100

$\boxed{0.6333}$