W08 - Examples

Simple divergence test

Simple divergence test: examples

Consider: $\displaystyle\sum_{n=1}^{\infty} \frac{n}{4n+1}$

- This diverges by the SDT because $a_n \to \frac{1}{4}$ and not 0.

Consider: $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n+1}$

- This diverges by the SDT because $\lim_{n\to\infty} a_n = \text{DNE}$.
- We can say the terms "converge to the pattern $+1,\ -1,\ +1,\ -1,\ \dots,$" but that is not a limit value.

Positive series

p-series examples

By finding p and applying the p-series convergence properties:

We see that $\sum_{n=1}^{\infty} \frac{1}{n^{1.1}}$ converges: $p = 1.1$ so $p > 1$

But $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges: $p = 1/2$ so $p \leq 1$

Integral test - pushing the envelope of convergence

Does $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ converge?

Does $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converge?

Notice that $\ln n$ grows *very slowly* with n, so $\frac{1}{n \ln n}$ is just a *little* smaller than $\frac{1}{n}$ for large n, and similarly $\frac{1}{n(\ln n)^2}$ is just a little smaller still.

Solution

1. The two series lead to the two functions $f(x) = \frac{1}{x \ln x}$ and $g(x) = \frac{1}{x(\ln x)^2}$.
2. Check applicability.
   - Clearly $f(x)$ and $g(x)$ are both continuous, positive, decreasing functions on $x \in [2, \infty]$.
3. Apply the integral test to $f(x)$.
   - Integrate $f(x)$:

$$\int_2^{\infty} \frac{1}{x \ln x}\, dx \quad \gg\gg \quad \int_{u=\ln 2}^{\infty} \frac{1}{u}\, du$$

$$\gg\gg \quad \lim_{R\to\infty} \ln u \Big|_{\ln 1}^{R} \quad \gg\gg \quad \infty$$

4. ≡ Conclude: $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ *diverges*.

5. ⇶ Apply the integral test to $g(x)$.

- Integrate $g(x)$:

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx \quad \gg\gg \quad \int_{u=\ln 2}^{\infty} \frac{1}{u^2}\,du$$

$$\gg\gg \quad \lim_{R\to\infty} -u^{-1}\Big|_{\ln 2}^{R} \quad \gg\gg \quad \frac{1}{\ln 2}$$

6. ≡ Conclude: $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ *converges*.

Direct comparison test: rational functions

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\,3^n}$ *converges* by the DCT.

- Choose: $a_n = \frac{1}{\sqrt{n}\,3^n}$ and $b_n = \frac{1}{3^n}$
- Check: $0 < \frac{1}{\sqrt{n}\,3^n} \leq \frac{1}{3^n}$
- Observe: $\sum \frac{1}{3^n}$ is a convergent geometric series

The series $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^3}$ *converges* by the DCT.

- Choose: $a_n = \frac{\cos^2 n}{n^3}$ and $b_n = \frac{1}{n^3}$.
- Check: $0 \leq \frac{\cos^2 n}{n^3} \leq \frac{1}{n^3}$
- Observe: $\sum \frac{1}{n^3}$ is a convergent p-series

The series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^3+1}$ *converges* by the DCT.

- Choose: $a_n = \frac{n}{n^3+1}$ and $b_n = \frac{1}{n^2}$
- Check: $0 \leq \frac{n}{n^3+1} \leq \frac{1}{n^2}$ (notice that $\frac{n}{n^3+1} \leq \frac{n}{n^3}$)
- Observe: $\sum \frac{1}{n^2}$ is a convergent p-series

The series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n-1}$ *diverges* by the DCT.

- Choose: $a_n = \frac{1}{n}$ and $b_n = \frac{1}{n-1}$
- Check: $0 \leq \frac{1}{n} \leq \frac{1}{n-1}$
- Observe: $\sum \frac{1}{n}$ is a divergent p-series

Limit comparison test examples

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ *converges* by the LCT.

- Choose: $a_n = \frac{1}{2^n-1}$ and $b_n = \frac{1}{2^n}$.
- Compare in the limit:

$$\lim_{n\to\infty} \frac{a_n}{b_n} \quad \gg\gg \quad \lim_{n\to\infty} \frac{2^n}{2^n-1} \quad \gg\gg \quad 1 =: L$$

- Observe: $\sum \frac{1}{2^n}$ is a convergent geometric series

The series $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$ *diverges* by the LCT.

- Choose: $a_n = \frac{2n^2+3n}{\sqrt{5+n^5}}$, $b_n = n^{-1/2}$
- Compare in the limit:

$$\lim_{n\to\infty} \frac{a_n}{b_n} \quad \gg\gg \quad \lim_{n\to\infty} \frac{(2n^2+3n)\sqrt{n}}{\sqrt{5+n^5}}$$

$$\frac{(2n^2+3n)\sqrt{n}}{\sqrt{5+n^5}} \quad \overset{n\to\infty}{\longrightarrow} \quad \frac{2n^{5/2}}{n^{5/2}} \to 2 \;=:\; L$$

- Observe: $\sum n^{-1/2}$ is a divergent p-series

The series $\displaystyle\sum_{n=2}^{\infty} \frac{n^2}{n^4-n-1}$ *converges* by the LCT.

- Choose: $a_n = \frac{n^2}{n^4-n-1}$ and $b_n = n^{-2}$
- Compare in the limit:

$$\lim_{n\to\infty} \frac{a_n}{b_n} \quad \gg\gg \quad \lim_{n\to\infty} \frac{n^4}{n^4-n-1} \quad \gg\gg \quad 1 \;=:\; L$$

- Observe: $\sum_{n=2}^{\infty} n^{-2}$ is a converging p-series

Alternating series

Alternating series test: basic illustration

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ converges by the AST.

- Notice that $\sum \frac{1}{\sqrt{n}}$ diverges as a p-series with $p = 1/2 < 1$.
- Therefore the first series converges *conditionally*.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2}$ converges by the AST.

- Notice the funny notation: $\cos n\pi = (-1)^n$.
- This series converges *absolutely* because $\left|\frac{\cos n\pi}{n^2}\right| = \frac{1}{n^2}$, which is a p-series with $p = 2 > 1$.

Approximating π

The Taylor series for $\tan^{-1} x$ is given by:

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

Use this series to approximate π with an error less than 0.001.

Solution

The main idea is to use $\tan\frac{\pi}{4} = 1$ and thus $\tan^{-1} 1 = \frac{\pi}{4}$. Therefore:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

and thus:

$$\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \cdots$$

Write E_n for the error of the approximation, meaning $E_n = S - S_n$.

By the AST error formula, we have $|E_n| < a_{n+1}$.

We desire n such that $|E_n| < 0.001$. Therefore, calculate n such that $a_{n+1} < 0.001$, and then we will know:

$$|E_n| < a_{n+1} < 0.001$$

The general term is $a_n = \frac{4}{2n-1}$. Plug in $n+1$ in place of n to find $a_{n+1} = \frac{4}{2n+1}$. Now solve:

$$a_{n+1} = \frac{4}{2n+1} < 0.001$$

$$\ggg \qquad \frac{4}{0.001} < 2n + 1$$

$$\ggg \qquad 3999 < 2n$$

$$\ggg \qquad 2000 \leq n$$

We conclude that at least 2000 terms are necessary to be confident (by the error formula) that the approximation of π is accurate to within 0.001.