W07 - Examples

Sequences

Geometric sequence: revealing the format

Find a_0 and r and a_n (written in the geometric sequence format) for the following geometric sequences:

(a) $a_n = \left(-\frac{1}{2}\right)^n$ (b) $b_n = -3\left(\frac{2^{n+1}}{5^n}\right)$ (c) $c_n = e^{5+7n}$

Solution

(a)

Plug in $n = 0$ to obtain $a_0 = 1$. Notice that $a_{n+1}/a_n = -1/2$ and so therefore $r = -1/2$. Then the 'general term' is $a_n = a_0 \cdot r^n = 1 \cdot (-1/2)^n$.

(b)

Rewrite the fraction:

$$\frac{2^{n+1}}{5^n} \quad \gg\gg \quad 2 \cdot \left(\frac{2}{5}\right)^n$$

Plug that in and observe $b_n = -6 \cdot (2/5)^n$. From this format we can *read off* $b_0 = -6$ and $r = 2/5$.

(c)

Rewrite:

$$c_n \quad \gg\gg \quad e^5 \cdot e^{7n} \quad \gg\gg \quad e^5 \cdot \left(e^7\right)^n$$

From this format we can *read off* $c_0 = e^5$ and $r = e^7$.

L'Hopital's Rule for sequence limits

(a) What is the limit of $a_n = \dfrac{\ln n}{n}$?

(b) What is the limit of $b_n = \dfrac{(\ln n)^2}{n}$?

(c) What is the limit of $c_n = n\left(\sqrt{n^2+1} - \sqrt{n}\right)$?

Solution

(a)

Identify indeterminate form $\frac{\infty}{\infty}$. Change from n to x and apply L'Hopital:

$$\lim_{x\to\infty} \frac{\ln x}{x} \quad \overset{\frac{d}{dx}}{\gg\gg} \quad \lim_{x\to\infty} \frac{1/x}{1} = 0$$

(b)

Identify indeterminate form $\frac{\infty}{\infty}$. Change from n to x and apply L'Hopital:

$$\lim_{x\to\infty} \frac{(\ln x)^2}{x} \quad \overset{\frac{d}{dx}}{\gg\gg} \quad \lim_{x\to\infty} \frac{2\ln x \cdot \frac{1}{x}}{1} = 2\lim_{x\to\infty} \frac{\ln x}{x} \quad \overset{\text{(by } a_n \text{ result)}}{=} \quad 0$$

(c)

Identify form $\infty \cdot 0$ and rewrite as $\frac{\infty}{\infty}$:

$$n\left(\sqrt{n^2+1}-\sqrt{n}\right) \qquad \gg\gg \qquad \frac{\sqrt{n^2+1}-\sqrt{n}}{1/n}$$

Change from n to x and apply L'Hopital:

$$\lim_{x\to\infty}\frac{\sqrt{x^2+1}-\sqrt{x}}{1/x} \qquad \gg\gg \qquad \frac{\frac{1}{2}\left(x^2+1\right)^{-1/2}(2x)-\frac{1}{2}x^{-1/2}}{-1/x^2}$$

Simplify:

$$\gg\gg \qquad \frac{-2x^3}{\sqrt{x^2+1}}+x^{3/2}=\frac{-2x^3+x^{3/2}\sqrt{x^2+1}}{\sqrt{x^2+1}}$$

Consider the limit:

$$\frac{-2x^3+x^{3/2}\sqrt{x^2+1}}{\sqrt{x^2+1}} \quad \overset{x\to\infty}{\longrightarrow} \quad \frac{-2x^3+x^{3/2}x}{x}\longrightarrow\frac{-2x^3}{x}\longrightarrow-\infty$$

Squeeze theorem

Use the squeeze theorem to show that $\frac{4^n}{n!}\to 0$ as $n\to\infty$.

Solution

We will squeeze the given general term above 0 and below a sequence b_n that we must devise:

$$0\le\frac{4^n}{n!}\le b_n$$

We need b_n to satisfy $b_n\to 0$ and $\frac{4^n}{n!}\le b_n$. Let us study $\frac{4^n}{n!}$.

$$\frac{4^n}{n!}=\frac{4\cdot 4\cdot\ \cdots\ \cdot 4\cdot 4\cdot 4\cdot 4\cdot 4\cdot 4\cdot 4}{n(n-1)\cdots 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}$$

Now for the trick. Collect factors in the middle bunch:

$$\frac{4^n}{n!}=\frac{4}{n}\left(\frac{4}{n-1}\cdot\frac{4}{n-2}\cdot\ \cdots\ \cdot\frac{4}{7}\cdot\frac{4}{6}\cdot\frac{4}{5}\right)\frac{4\cdot 4\cdot 4\cdot 4}{4\cdot 3\cdot 2\cdot 1}$$

Each factor in the middle bunch is <1 so the entire middle bunch is <1. Therefore:

$$\frac{4^n}{n!}<\frac{4}{n}\cdot\frac{4^4}{4!}=\frac{1024}{24n}$$

Now we can easily see that $1024/24n\to 0$ as $n\to\infty$, so we set $b_n=1024/24n$ and we are done.

Monotonicity

Show that $a_n=\sqrt{n+1}-\sqrt{n}$ converges.

Solution

1. Observe that $a_n>0$ for all n.
   - Because $n+1>n$, we know $\sqrt{n+1}>\sqrt{n}$.
   - Therefore $\sqrt{n+1}-\sqrt{n}>0$
2. Change n to x and show a_x is decreasing.
   - New formula: $a_x=\sqrt{x+1}-\sqrt{x}$ considered as a *differentiable* function.
   - ⚠ Take derivative to show decreasing.

- Derivative of a_x:

$$\frac{d}{dx}a_x = \frac{1}{2\sqrt{x+1}} - \frac{1}{2\sqrt{x}}$$

- Simplify:

$$\gg\gg \qquad \frac{2\left(\sqrt{x}-\sqrt{x+1}\right)}{4\sqrt{x}\sqrt{x+1}}$$

- Denominator is > 0. Numerator is < 0. So $\frac{d}{dx}a_x < 0$ and a_x is monotone decreasing.

3. ≡ Therefore a_n is monotone decreasing as $n \to \infty$.

Series

Geometric series - total sum and partial sums

The geometric series *total sum S* can be calculated using a *"shift technique"* as follows:

1. Compare S and rS:

$$\begin{aligned} S &= a_0 + a_0r + a_0r^2 + a_0r^3 + \cdots \\ \overset{\times r}{\gg\gg} \qquad rS &= a_0r + a_0r^2 + a_0r^3 + a_0r^4 + \cdots \end{aligned}$$

2. Subtract second line from first line, many cancellations:

$$\begin{aligned} S &= a_0 + a_0r + a_0r^2 + a_0r^3 + \cdots \\ -\Big(rS &= a_0r + a_0r^2 + a_0r^3 + a_0r^4 + \cdots\Big) \\ \hline S - rS &= a_0 \end{aligned}$$

3. Solve to find S:

$$S = \frac{a_0}{1-r}$$

- ⚠ Note: this calculation *assumes* that S exists, i.e. that the series *converges*.

The geometric series *partial sums* can be calculated similarly, as follows:

1. Compare S and rS:

$$\begin{aligned} S_N &= a_0 + a_0r + a_0r^2 + \cdots + a_0r^N \\ \overset{\times r}{\gg\gg} \qquad rS_N &= a_0r + a_0r^2 + \cdots + a_0r^N + a_0r^{N+1} \end{aligned}$$

2. Subtract second line from first line, many cancellations:

$$\begin{aligned} S_N &= a_0 + a_0r + a_0r^2 + \cdots + a_0r^N \\ -\Big(rS_N &= a_0r + a_0r^2 + \cdots + a_0r^N + a_0r^{N+1}\Big) \\ \hline S_N - rS_N &= a_0 - a_0r^{N+1} \end{aligned}$$

3. Solve to find S_N:

$$S_N = a_0 \frac{1 - r^{N+1}}{1 - r}$$

$$= \frac{a_0}{1-r} - \frac{a_0}{1-r} r^{N+1} = S - S r^{N+1}$$

- The last formula is revealing in its own way. Here is what it means in terms of terms:

$$a_0 + a_0 r + \cdots + a_0 r^N =$$

$$a_0 + a_0 r + a_0 r^2 + \cdots$$
$$- \left(a_0 r^{N+1} + a_0 r^{N+2} + \cdots \right)$$