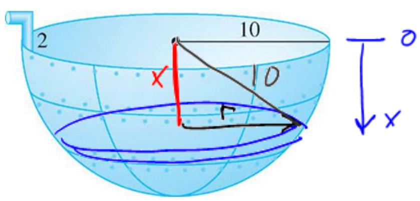
Name: Solutions

Worksheet 6.4a – Work

1) a) Set up an integral to calculate the work required to pump all of the water out of the tank, which is initially full. Distances are in meters, and the water exits the spigot shown. (LT: 2b)



$d = x + 2$

$r^2 + x^2 = 10^2$

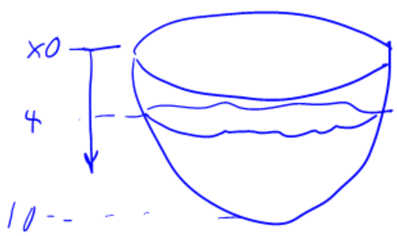
$r^2 = 100 - x^2$

$$W_i = \rho g V_i d_i$$
$$= \rho g \pi r_i^2 \Delta x (x_i + 2)$$
$$= \rho g \pi (100 - x_i^2) \Delta x (x_i + 2)$$

$$W = \int_0^{10} \rho g \pi (100 - x^2)(x + 2)\, dx$$

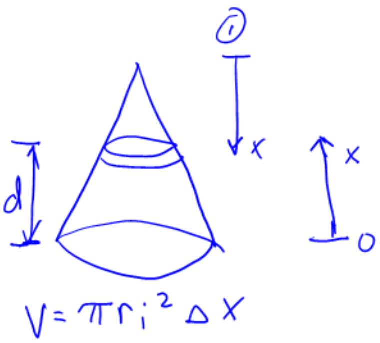
$$W = \boxed{\int_0^{10} \rho g \pi (100 - x^2)(x + 2)\, dx}$$

b) Set up an integral to calculate the work required to pump the water out of the tank if it is initially full to a depth of 6 m. (LT: 2b)

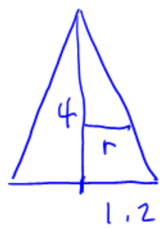


$$W = \boxed{\int_4^{10} \rho g \pi (100 - x^2)(x + 2)\, dx}$$

2) Set up an integral to calculate the work against gravity to build a right circular cone of height 4 m and base radius 1.2 m out of a lightweight material of density 600 kg/m^3. (LT: 2b)



$V = \pi r_i^2 \Delta x$



option 1:

$d = 4 - x$

Using similar triangles:

$$\frac{x}{4} = \frac{r}{1.2}$$

$$r = \frac{1.2}{4} x = 0.3x$$

$$W = \int_0^4 \rho g \pi (0.3x)^2 (4-x)\,dx$$

option 2:

$d = x$

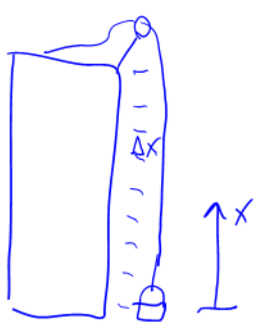
Using similar triangles:

$$\frac{4-x}{4} = \frac{r}{1.2}$$

$$r = \frac{1.2}{4}(4-x) = 0.3(4-x)$$

$$W = \int_0^4 \rho g \pi (0.3(4-x))^2 x\,dx = \int_0^4 \rho g \pi (1.2 - 0.3x)^2 x\,dx$$

3) What is the work required to lift a bucket of water to the top of a 50-foot building if the bucket weighs 4 lbs and initially contains 30 lbs of water? The chain weighs 0.25 lbs per foot and the water is leaking at a rate of 0.3 lbs per second as the bucket is lifted at a constant rate of 3 feet per second. (LT: 2b)



$W_i = F_i d_i$

$F_i = F_B + F_w + F_c \qquad d_i = \Delta x$

$F_B = 4 \qquad F_w = 30 - \frac{0.3}{3} x_i \qquad F_c = 50(0.25) - 0.25 x_i$

$W_i = (4 + (30 - 0.1x_i) + (12.5 - 0.25x_i))\,\Delta x$

$W_i = (46.5 - 0.35x_i)\,\Delta x$

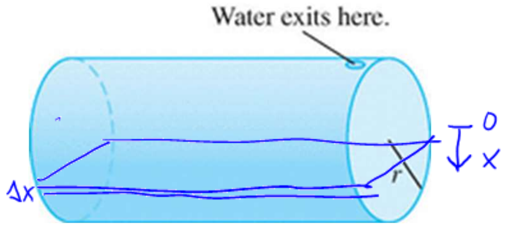
$$W = \int_0^{50} (46.5 - 0.35x)\,dx = \left(46.5x - \frac{0.35x^2}{2}\right)\Bigg|_0^{50} = 1887.5 \text{ ft-lb}$$

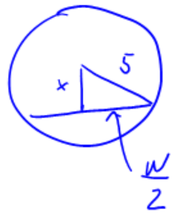
$\boxed{1887.5 \text{ ft-lb}}$

Name: Solutions

Worksheet 6.4b – Work

1) The horizontal cylindrical tank shown has a radius of r = 5m and a length of 7m.





$$\left(\frac{w}{2}\right)^2 + x^2 = 5^2$$
$$w = 2\sqrt{25 - x^2}$$

a) Set up an integral to calculate the work required to pump all of the water out of the tank, which is initially full. Assume that the water exits from a small hole at the top. (LT: 2b)

distance $d = x + 5$

$$W_i = \rho g V_i d_i$$
$$V_i = 7\left(2\sqrt{25 - x_i^2}\right)\Delta x$$

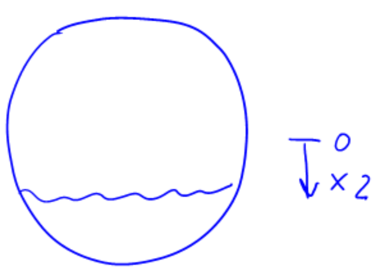
$$\boxed{W = \int_{-5}^{5} \rho g (7)\left(2\sqrt{25 - x^2}\right)(x+5)\, dx}$$

b) Set up an integral to calculate the work required to pump the water out of the tank if it is initially full to a depth of 3 m, and the water exits through a spigot that is 1 m above the tank. (LT: 2b)

distance $d = x + 5 + 1 = x + 6$

$$W = \boxed{\int_{2}^{5} \rho g (7) 2\sqrt{25 - x^2}\,(x+6)\, dx}$$

c) Set up an integral to calculate the fluid force on the front face of the tank in part b. (LT: 2a)



$$F = \boxed{\int_{2}^{5} \rho g (x-2)\left(2\sqrt{25 - x^2}\right) dx}$$

depth $d = x - 2$

$$F_i = P_i A_i = \rho g d_i w_i \Delta x$$
$$= \rho g (x_i - 2)\left(2\sqrt{25 - x_i^2}\right)\Delta x$$

2) Built around 2600 BC, the Great Pyramid of Giza in Egypt is 146 m high and has a square base of side 230 m. set up an integral to find the work (against gravity) required to build the pyramid if the density of the stone is estimated at 2000 kg/m^3. (LT: 2a)

1: $\uparrow x$

2: $\downarrow x$

$V_i = S_i^{\,2} \Delta x$

option 1

(x, s), $(0, 230)$, $(146, 0)$

$s - 230 = \frac{-230}{146}(x - 0)$

$s = 230 - \frac{230}{146}x$

$d = x$

option 2

(x, s), $(0, 0)$, $(146, 230)$

$s - 0 = \frac{230}{146}(x - 0)$

$s = \frac{230}{146}x$

$d = 146 - x$

option 1) $W = \int_0^{146} \rho g \left(230 - \frac{230}{146}x\right)^2 x \, dx$

option 2) $W = \int_0^{146} \rho g \left(\frac{230}{146}x\right)^2 (146 - x) \, dx$

Name: Solutions

Worksheet 7.8a – Improper Integrals

1) Which of the following integrals is improper? Next to each integral write P for Proper or write I for Improper due to an infinite interval or D for Improper due to a discontinuous integrand. (LT: 3)

a) $\int_0^2 \frac{dx}{x^{1/3}}$	D (0)		d) $\int_0^\infty \sin x dx$	I
b) $\int_0^1 e^{-x} dx$	P		e) $\int_1^\infty \ln x dx$	I
c) $\int_0^{\pi/2} \sec x dx$	D $(\frac{\pi}{2})$		f) $\int_0^3 \ln x dx$	D (0)

2) First predict whether the integral will be convergent or divergent. Then evaluate. (Find the limit and circle Convergent or Divergent.) (LT: 3)

a) $\int_1^\infty \frac{dx}{x^{19/20}} = \lim_{R\to\infty} \int_1^R x^{-\frac{19}{20}} dx$

$= \lim_{R\to\infty} 20x^{\frac{1}{20}} \Big|_1^R$

$= \lim_{R\to\infty} 20(R^{\frac{1}{20}} - 1)$

$= \infty$

Limit: ∞

Convergent/Divergent

b) $\int_0^5 \frac{dx}{x^{20/19}} = \lim_{R\to0^+} \int_R^5 x^{-\frac{20}{19}} dx$

$= \lim_{R\to0^+} -19x^{-\frac{1}{19}} \Big|_R^5$

$= \lim_{R\to0^+} -19(5^{-\frac{1}{19}} - R^{-\frac{1}{19}})$

$= \infty$

Limit: ∞

Convergent/Divergent

c) $\int_1^\infty e^{-x} dx = \lim_{R\to\infty} \int_1^R e^{-x} dx$

$= \lim_{R\to\infty} -e^{-x} \Big|_1^R$

$= \lim_{R\to\infty} (-e^{-R} + e^{-1})$

$= e^{-1}$

Limit: e^{-1}

Convergent/Divergent

d) $\int_0^1 \ln x\,dx = \lim_{R\to 0^+} \int_R^1 \ln x\,dx$

$= \lim_{R\to 0^+} (x\ln x - x)\Big|_R^1$

$= \lim_{R\to 0^+} (1\ln 1 - 1 - R\ln R + R)$

$= -1$

$\lim_{R\to 0^+} R\ln R$ type $0\cdot(-\infty)$

$= \lim_{R\to 0^+} \frac{\ln R}{\left(\frac{1}{R}\right)}$ type $\frac{-\infty}{\infty}$

$\overset{H}{=} \lim_{R\to 0^+} \frac{\left(\frac{1}{R}\right)}{\left(-\frac{1}{R^2}\right)}$

$= \lim_{R\to\infty} -R$

$= 0$

Limit: -1

Convergent/Divergent

e) $\int_{-\infty}^{\infty} xe^{-x^2}dx = \int_{-\infty}^{0} xe^{-x^2}dx + \int_0^{\infty} xe^{-x^2}dx$

$\int_0^{\infty} xe^{-x^2}dx = \lim_{R\to\infty} \int_0^R xe^{-x^2}dx$

$u = -x^2$

$du = -2x\,dx$

$= \lim_{R\to\infty} \int_0^{-R^2} -\frac{1}{2}e^u\,du$

$= \lim_{R\to\infty} -\frac{1}{2}e^u\Big|_0^{R^2}$

$= \lim_{R\to\infty} \left(-\frac{1}{2}e^{-R^2} + \frac{1}{2}\right)$

$= \frac{1}{2}$

$\int_{-\infty}^{0} xe^{-x^2}dx = -\frac{1}{2}$ (by odd symmetry)

$\int_{-\infty}^{\infty} xe^{x^2}dx = 0$

Limit: 0

Convergent/Divergent

Supplement: Predict whether the following integral will converge or diverge. Use the Comparison Test to make your prediction. (LT: 3)

$$\int_1^{\infty} \frac{1}{x^4+\sqrt{x}}dx$$

Convergent/Divergent:

$0 < \frac{1}{x^4+\sqrt{x}} < \frac{1}{x^4}$

$\int_1^{\infty} \frac{1}{x^4}dx$ is convergent $(p=4>1)$

so $\int_1^{\infty} \frac{1}{x^4+\sqrt{x}}dx$ is convergent by Comparison Test

Name: Solutions

Worksheet 7.8b – Improper Integrals

1) The Laplace Transform of a function, *f(x)*, is the function *Lf(s)* of the variable *s* defined by the improper integral

$$Lf(s)=\int_0^\infty f(x)e^{-sx}dx$$

if the integral converges. (LT: 3)

a) Show that if $f(x)=C$, where C is a constant, then $Lf(s)=\frac{C}{s}$ for s > 0.

$$Lf(s)=\int_0^\infty Ce^{-sx}dx$$

$$=\lim_{R\to\infty}\int_0^R Ce^{-sx}dx$$

$$=\lim_{R\to\infty}\left[-\frac{C}{s}e^{-sx}\right]_0^R$$

$$=\lim_{R\to\infty}\left[-\frac{C}{s}\left(e^{-sR}-1\right)\right]$$

$$=\frac{C}{s} \quad \text{if } s>0$$

$$Lf(s)=\frac{C}{s} \quad \checkmark$$

b) Compute $Lf(s)$, where $f(x)=e^x$ and s > 1.

$$Lf(s)=\int_0^\infty e^x e^{-sx}dx$$

$$=\lim_{R\to\infty}\int_0^R e^{(1-s)x}dx$$

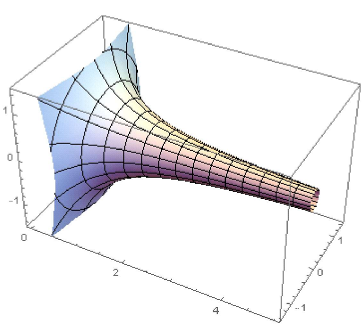
$$=\lim_{R\to\infty}\left.\frac{e^{(1-s)x}}{1-s}\right|_0^R$$

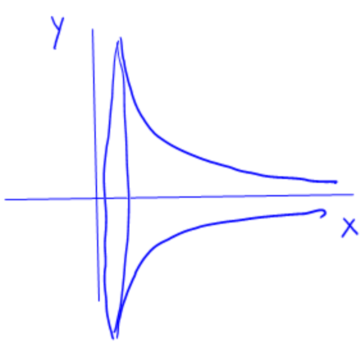
$$=\lim_{R\to\infty}\frac{1}{1-s}\left(e^{(1-s)R}-1\right)$$

$$=-\frac{1}{1-s} \quad \text{if } s>1$$

$$\boxed{Lf(s)=\frac{1}{s-1}}$$

Supplement: The solid region G obtained by rotating the region below the graph of $y=x^{-1}$ about the x-axis for $1\le x\le\infty$ is called Gabriel's Horn. Show that the volume of G is finite and that the surface area of G is infinite. (LT: 3)





$$V_G=\int_1^\infty \pi\left(\frac{1}{x}\right)^2dx$$

$$=\lim_{R\to\infty}\int_1^R \pi x^{-2}dx$$

$$=\lim_{R\to\infty}\left.-\frac{\pi}{x}\right|_1^R$$

$$=\lim_{R\to\infty}\pi\left(-\frac{1}{R}+1\right)$$

$$=\pi \quad \text{(finite}$$

$$S_G=\int_1^\infty 2\pi\left(\frac{1}{x}\right)\sqrt{1+\left(-\frac{1}{x^2}\right)^2}\,dx$$

$$=2\pi\int_1^\infty \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx$$

$$\frac{1}{x}\sqrt{1+\frac{1}{x^4}}>\frac{1}{x}>0$$

$\int_1^\infty \frac{1}{x}dx$ is divergent (∞)
($p=1 \not> 1$)

$\int_1^\infty \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx$ is divergent (∞) by Comparison Test

S_G is infinite