W06 - Examples

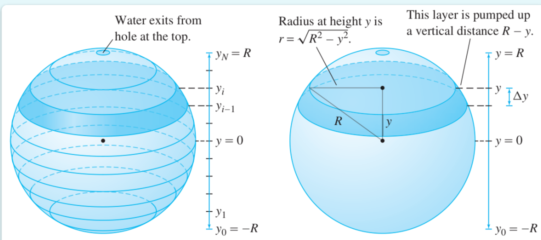
Work performed

Pumping water from spherical tank

Calculate the work done pumping water out of a spherical tank of radius 5 m.

Solution

1. ☰ Divide the sphere of water into horizontal layers.



- Coordinate y is $y = 0$ at the center of the sphere.

2. ☰ Work done pumping out water is constant across any single layer.
3. ☰ Find formula for weight of a single layer.
   - Area of the layer at y is $A(y) = \pi(5^2 - y^2)$ because its radius is $r = \sqrt{5^2 - y^2}$.
   - Volume of the layer at y is then $\pi(5^2 - y^2)\,dy$.
   - Weight of the layer is then $F(y)\,dy = g \cdot \rho \cdot \pi(5^2 - y^2)\,dy$.
     - Plug in:

$$g = 9.8\,\frac{m}{s^2}, \quad \rho = 1000\,\frac{kg}{m^3} \qquad \gg\gg \qquad F(y)\,dy = \left(9800\,\frac{kg}{m^2 s^2}\right)\pi(5^2 - y^2)\,dy$$

4. ☰ Find formula for vertical distance a given layer is lifted.
   - Layer at y must be lifted by $5 - y$ to the top of the tank.
5. ☰ Work per layer is the product.
   - Product of weight times height lifted:

$$\left(9800\,\frac{kg}{m^2 s^2}\right)\pi(5^2 - y^2)(5 - y)\,dy$$

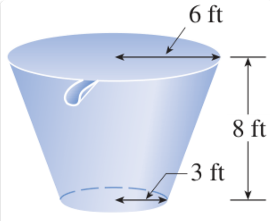
6. ☰ Total work done is the integral.
   - Integrate over the layers:

$$W = \int_{-5}^{+5} \left(9800\,\frac{kg}{m^2 s^2}\right)\pi(5^2 - y^2)(5 - y)\,dy \quad \approx \quad 2.6 \times 10^7\,\mathrm{J}$$

7. ⚠ Supplement: what if the spigot sits 2m above the tank?
   - Increase the height function from $5 - y$ to $7 - y$.
8. ⚠ Supplement: what if the tank starts at just 3m of water depth?
   - Integrate the water layers only: change bounds to $\int_{-5}^{-2}$.

Water pumped from a frustum

Find the work required to pump water out of the frustum in the figure. Assume the weight of water is $\rho = 62.5\,\mathrm{lb}/\mathrm{ft}^3$.



frustum of a cone

Solution

1. Find weight of a horizontal slice.
    - Coordinate $y = 0$ at top, increasing downwards.
    - Use $r(y)$ for radius of cross-section circle.
    - Linear decrease in r from $r(0) = 6$ to $r(8) = 3$:

$$r(y) = 6 - \frac{3}{8}y$$

    - Area is πr^2:

$$\text{Area}(y) = \pi\left(6 - \tfrac{3}{8}y\right)^2$$

    - Weight $=$ density $\times$ area $\times$ thickness:

$$\text{weight of layer} = \rho\pi\left(6 - \tfrac{3}{8}y\right)^2\,dy$$

2. Find work to pump out a horizontal layer.
    - Layer at y is raised a distance of y.
    - Work to raise layer at y:

$$\rho\pi y\left(6 - \tfrac{3}{8}y\right)^2\,dy$$

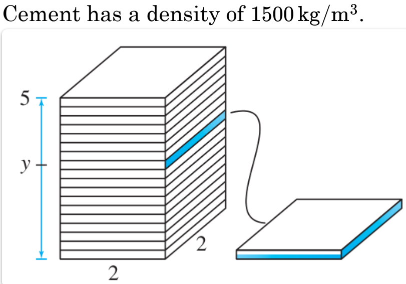
3. Integrate over all layers.
    - Integrate from top to bottom of frustum:

$$\begin{aligned}\int_0^8 \rho\pi y\left(6 - \tfrac{3}{8}y\right)^2\,dy &= 528\pi\rho \\ &= 528\pi\cdot 62.5 \\ &\approx 1.04\times 10^5\ \text{ft-lb}\end{aligned}$$

    - Final answer is 1.04×10^5 ft-lb.

Raising a building

Find the work done to raise a cement columnar building of height $5\,\text{m}$ and square base $2\,\text{m}$ per side. Cement has a density of $1500\,\text{kg/m}^3$.



Solution

1. Divide the building into horizontal layers.
    - Work done raising up the layers is constant for each layer.
2. Find formula for weight of each layer.
    - Volume $=$ area $\times$ thickness $= 4\,dy$
    - Mass $=$ density $\times$ volume $= 1500 \times 4\,dy = 6000\,dy$

- Weight of layer $= g \times \text{mass} = 9.8 \times 6000\, dy = 58800\, dy$

3. Find formula for work performed lifting one layer into place.
   - Work $=$ weight $\times$ distance lifted $= 58800 \times y\, dy$
4. Find total work as integral over the layers.
   - Total work $= \int_0^5 58000y\, dy = 735\,\text{kJ}$

Raising a chain

An $80\,\text{ft}$ chain is suspended from the top of a building. Suppose the chain has weight density $0.5\,\text{lb/ft}$. What is the total work required to reel in the chain?

Solution

1. Divide the chain into horizontal layers.
   - Each layer has vertical thickness dy.
   - Each layer has weight $0.5\, dy$ in lb.
2. Find formula for distance each layer is raised.
   - Each layer is raised from y to $80\,\text{ft}$, a distance of $(80 - y)\,\text{ft}$.
3. Compute total work.
   - Work to raise each layer is weight times distance raised:

$$\text{Work to raise layer} = (80 - y) \cdot 0.5\, dy$$

   - Add the work over all layers:

$$\int_0^{80} (80 - y) \cdot 0.5\, dy = 1600\,\text{ft-lb}$$

Raising a leaky bucket

Suppose a bucket is hoisted by a cable up an $80\,\text{ft}$ tower. The bucket is lifted at a constant rate of $2\,\text{ft/sec}$ and is leaking water weight at a constant rate of $0.2\,\text{lb/sec}$. The initial weight of water is $50\,\text{lb}$. What is the total work performed against gravity in lifting the water? (Ignore the bucket itself and the cable.)

Solution

1. Convert to static format.
   - Compute rate of water weight loss per unit of vertical height:

$$\frac{\text{rate of leak}}{\text{rate of lift}} = \text{leaked weight per foot}, \qquad \frac{0.2\,\text{lb/sec}}{2\,\text{ft/sec}} = 0.1\,\text{lb/ft}$$

   - Choose coordinate $y = 0$ at base, $y = 80$ at top.
   - Compute water weight at each height y:

$$\text{water weight} = (50 - 0.1y)\,\text{lb}$$

2. ⚠ Work formula.
   - Total work is integral of force times infinitesimal distance:

$$\text{work} = \int_a^b F(y)\, dy$$

3. Integrate weight times dy.
   - Plug in weight as force:

$$\text{work} = \int_0^{80} (50 - 0.1y)\, dy$$

   - Compute integral:

$$\gg\gg \qquad 50y - 0.05y^2\Big|_0^{80} = 3680\,\text{ft-lb}$$

Improper integrals

Improper integral - infinite bound

Show that the improper integral $\int_2^\infty \frac{dx}{x^3}$ converges. What is its value?

Solution

1. ≡ Replace infinity with a new symbol R.
   - Compute the integral:

$$\int_2^R \frac{dx}{x^3} = -\frac{1}{2}x^{-2}\Big|_2^R = \frac{1}{8} - \frac{1}{2R^2}$$

2. ≡ Take limit as $R \to \infty$.
   - Find limit:

$$\lim_{R\to\infty} \frac{1}{8} - \frac{1}{2R^2} = \frac{1}{8}$$

3. ≡ Apply definition of improper integral.
   - By definition:

$$\int_2^\infty \frac{dx}{x^3} = \lim_{R\to\infty} \int_2^R \frac{dx}{x^3} = \frac{1}{8}$$

4. ≡ Conclude that $\int_2^\infty \frac{dx}{x^3}$ converges and equals $\frac{1}{8}$.

Improper integral - infinite integrand

Show that the improper integral $\int_0^9 \frac{dx}{\sqrt{x}}$ converges. What is its value?

Solution

1. ≡ Replace the 0 where $\frac{1}{\sqrt{x}}$ diverges with a new symbol a.
   - Compute the integral:

$$\int_a^9 \frac{dx}{\sqrt{x}} = \int_a^9 x^{-1/2}\,dx = 2x^{+1/2}\Big|_a^9 = 6 - 2\sqrt{a}$$

2. ≡ Take limit as $a \to 0^+$.
   - Find limit:

$$\lim_{a\to 0^+} 6 - 2\sqrt{a} = 6$$

3. ≡ Apply definition of improper integral.
   - By definition:

$$\int_a^9 \frac{dx}{\sqrt{x}} = \lim_{a\to 0^+} \int_a^9 \frac{dx}{\sqrt{x}} = 6$$

4. ≡ Conclude that $\int_0^9 \frac{dx}{\sqrt{x}}$ converges to 6.

Example - Improper integral - infinity inside the interval

Does the integral $\int_{-1}^{+1} \frac{1}{x}\,dx$ converge or diverge?

Solution

It is *tempting* to compute the integral *incorrectly*, like this:

$$\int_{-1}^{+1} \frac{1}{x}\,dx = \ln|x|\Big|_{-1}^{+1} = \ln|2| - \ln|-2| = 0$$

But this is wrong. There is an infinite integrand at $x = 0$. We must instead break it into parts.

1. ⚠ Identify infinite integrand at $x = 0$.

- Integral becomes:

$$\int_{-1}^{+1} \frac{1}{x}\, dx = \int_{-1}^{0} \frac{1}{x}\, dx + \int_{0}^{+1} \frac{1}{x}\, dx$$

2. ≡ Interpret improper integrals.
   - Limit interpretations:

$$\lim_{R \to 0^-} \int_{-1}^{R} \frac{1}{x}\, dx + \lim_{R \to 0^+} \int_{R}^{+1} \frac{1}{x}\, dx$$

3. ≡ Compute integrals.
   - Using $\int \frac{1}{x}\, dx = \ln |x| + C$:

$$\int_{-1}^{R} \frac{1}{x}\, dx = \ln |R| - \ln |-1| = \ln |R|, \qquad \int_{R}^{+1} \frac{1}{x}\, dx = \ln |1| - \ln |R| = -\ln R$$

4. ≡ Take limits.
   - We have:

$$\lim_{R \to 0^-} \ln |R| = -\infty, \qquad \lim_{R \to 0^+} -\ln R = +\infty$$

   - *Neither* limit is finite. For $\int_{-1}^{+1} \frac{1}{x}\, dx$ to exist we'd need *both* of these limits to be finite.

Comparison to p-integrals

Determine whether the integral converges:

- (a) $\int_2^\infty \frac{x^3}{x^4-1}\, dx$
- (b) $\int_1^\infty \frac{1}{x^2+x+1}\, dx$

Solution

(a)

1. ⚠ Integrand tends toward $1/x$ for large x.
   - Consider large x values:

$$\frac{x^3}{x^4-1} \quad \longrightarrow \quad \frac{x^3}{x^4} \quad \text{for} \quad x \to \infty, \qquad \text{and} \quad \frac{x^3}{x^4} = \frac{1}{x}$$

2. ≡ Try comparison to $1/x$.
   - Comparison attempt:

$$\frac{x^3}{x^4-1} \overset{?}{>} \frac{1}{x}$$

   - Validate. Notice $x^4 - 1 > 0$ and $x > 0$ when $x \geq 2$.

$$\frac{x^3}{x^4-1} \overset{?}{>} \frac{1}{x} \qquad \gg\!\gg \qquad x^3 \cdot x \overset{?}{>} 1 \cdot (x^4 - 1) \qquad \gg\!\gg \qquad x^4 \overset{\checkmark}{>} x^4 - 1$$

3. ≡ Apply comparison test.
   - We know:

$$\frac{x^3}{x^4-1} > \frac{1}{x}$$

   - We know:

$$\int_2^\infty \frac{1}{x}\, dx \qquad \text{diverges}$$

   - We conclude:

$$\int_2^\infty \frac{x^3}{x^4-1}\, dx \qquad \text{diverges}$$

(b)

1. ⚠ Integrand tends toward $1/x^2$ for large x.
   - Consider large x values:

$$\frac{1}{x^2+x+1} \quad \longrightarrow \quad \frac{1}{x^2} \quad \text{for} \quad x \to \infty$$

2. ≡ Try comparison to $1/x^2$.
   - Comparison attempt:

$$\frac{1}{x^2+x+1} \stackrel{?}{<} \frac{1}{x^2}$$

   - Validate. Notice $x^2+x+1>0$ and $x^2>0$ when $x \geq 1$.

$$\frac{1}{x^2+x+1} \stackrel{?}{<} \frac{1}{x^2} \qquad \gg\gg \qquad 1 \cdot x^2 \stackrel{?}{<} 1 \cdot (x^2+x+1) \qquad \gg\gg \qquad x^2 \stackrel{\checkmark}{<} x^2+x+1$$

3. Apply comparison test.
   - We know:

$$\frac{1}{x^2+x+1} < \frac{1}{x^2}$$

   - We know:

$$\int_1^\infty \frac{1}{x^2}\,dx \qquad \text{converges}$$

   - We conclude:

$$\int_1^\infty \frac{1}{x^2+x+1}\,dx \qquad \text{converges}$$