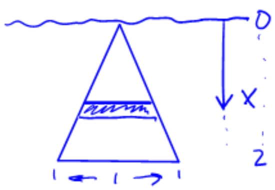
Name: Solutions

Worksheet 8.3a – Hydrostatic Pressure and Force

1) A plate in the shape of an isosceles triangle with base 1m and height 2m is submerged vertically in a tank of water so that its vertex touches the surface of the water. Set up an integral to find the fluid force on one side of the plate. (LT: 2a)



$F_i = P_i A_i$
$= (\rho g d_i)(w_i \, \Delta x)$
$= (\rho g x_i)(\frac{1}{2} x_i \, \Delta x)$

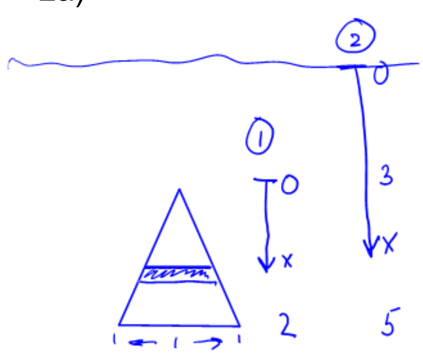
$d_i = x_i$

(x, w)
$(0, 0)$
$(2, 1)$

$w - 0 = \frac{1}{2}(x - 0)$
$w = \frac{1}{2} x$

$$\boxed{F = \int_0^2 \rho g x \left(\frac{1}{2} x\right) dx}$$

2) Repeat exercise 1, but assume that the vertex is located 3m below the surface of the water. (LT: 2a)



option 1) $d_i = x_i + 3$
$w_i = \frac{1}{2} x_i$ (as above)

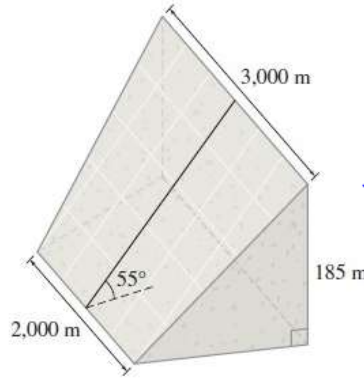
$$F = \int_0^2 \rho g (x+3)\left(\frac{1}{2} x\right) dx$$

option 2) $d_i = x_i$
$w_i = \frac{1}{2}(x_i - 3)$

(x, w)
$(3, 0)$
$(5, 1)$
$w - 0 = \frac{1}{2}(x - 3)$
$w = \frac{1}{2}(x - 3)$

$$F = \int_3^5 \rho g x \left(\frac{1}{2}(x-3)\right) dx$$

3) The Three Gorges Dam on China's Yangtze River has height 185 m. Set up an integral to calculate the force on the dam assuming that the dam is a trapezoid of base 2000 m and upper edge 3000 m, inclined at an angle of 55º to the horizontal. Do not evaluate the integral. (LT: 2a)



0, $\downarrow x$

(x, w)
$(0, 3000)$
$(185, 2000)$

$w - 3000 = -\frac{1000}{85}(x - 0)$

185

$w = 300 - \frac{1000}{185}x$

$d = x_i$

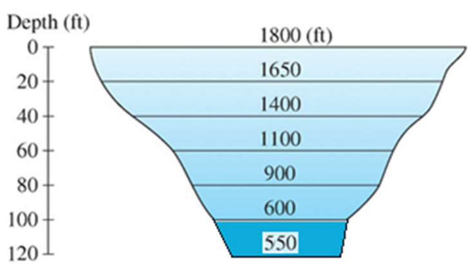
$$F_i = P_i A_i$$
$$= (\rho g d_i)(w_i)\left(\frac{1}{\sin 55^\circ}\Delta x\right)$$
$$= \rho g x_i \left(3000 - \frac{1000}{85}x_i\right)\left(\frac{1}{\sin 55^\circ}\right)\Delta x$$

$$F = \boxed{\int_0^{185} \frac{1}{\sin 55^\circ}(\rho g)\, x\left(3000 - \frac{1000}{185}x\right)dx}$$

4) The figure shows the wall of a dam on a water reservoir. Use Simpson's Rule and the width and depth measurements in the figure to estimate the fluid force on the wall. (LT: 2a, 1i)

$d = x$
$w = w(x)$ (function of x)

$x \downarrow$

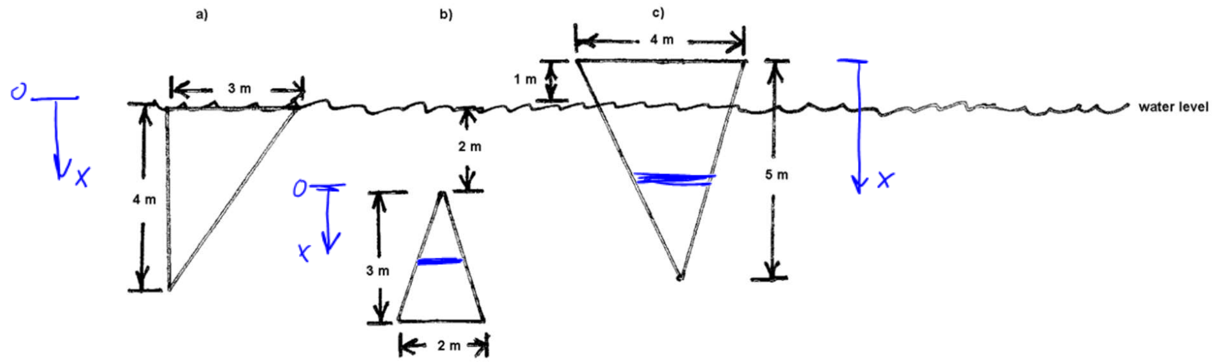


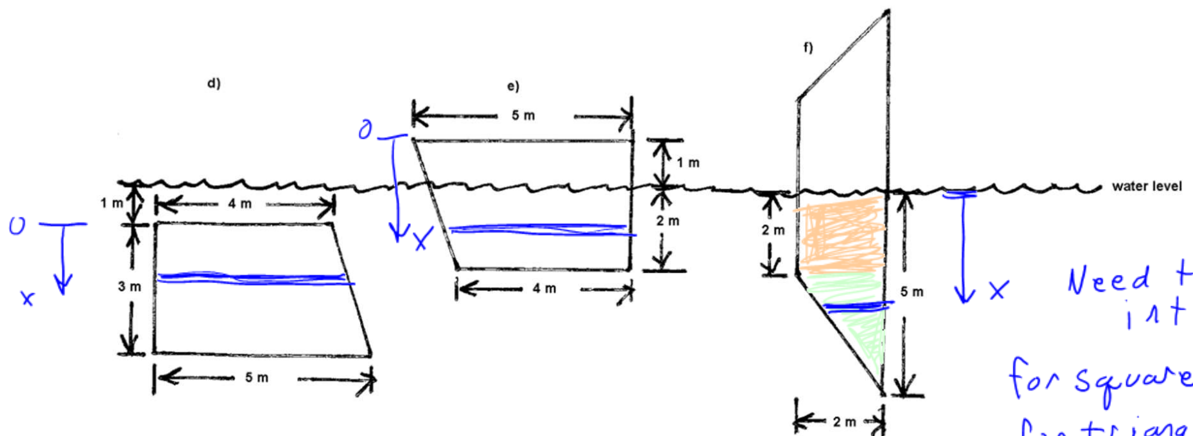
$$F = \int_0^{120} \delta \; x\, w(x)\, dx \qquad \delta = 62.5 \frac{lb}{ft^3}$$

$$F \approx S_6 = \frac{\delta \Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6) \quad \text{where } y_i = x_i w(x_i)$$
$$= \frac{20}{3}(\delta)\big(0(1800) + 4(20)(1650) + 2(40)(1400) + 4(60)(1100) + 2(80)(900) + 4(100)(600) + 120(550)\big)$$
$$=$$

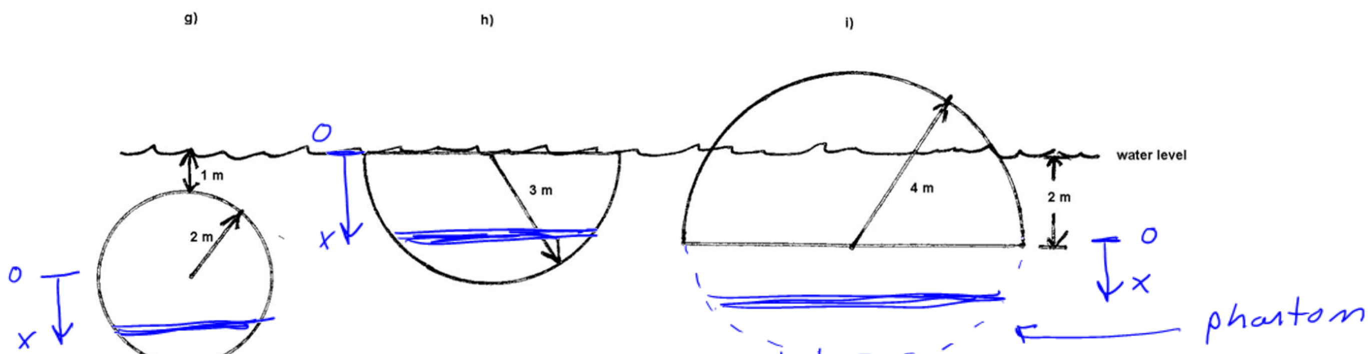
$$F \approx \boxed{3.99 \times 10^8 \text{ lb}}$$

Supplement: For each of the plates shown, a-i, set up an integral to compute the fluid force on one face of the plate which is submerged in water. (LT: 2a)





Need two integrals
for square $w=2$
for triangle $w=2-\frac{2}{3}(x-2)$



Other answers are possible!

a) $\int_0^4 \rho g x(3-\frac{3}{4}x)\,dx$	b) $\int_0^3 \rho g (x+2)(\frac{2}{3}x)\,dx$	c) $\int_1^5 \rho g (x-1)(4-\frac{4}{5}x)\,dx$
d) $\int_0^3 \rho g (x+1)(4+\frac{1}{3}x)\,dx$	e) $\int_1^3 \rho g (x-1)(5-\frac{1}{3}x)\,dx$	f) $\int_0^2 \rho g x(2)\,dx + \int_2^5 \rho g x(2-\frac{2}{3}(x-2))\,dx$
g) $\int_{-2}^2 \rho g (x+3)(2\sqrt{4-x^2})\,dx$	h) $\int_0^3 \rho g x(2\sqrt{9-x^2})\,dx$	i) $\int_{-2}^0 \rho g (x+2)(2\sqrt{16-x^2})\,dx$

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Worksheet 8.3b.1 – Moments and Center of Mass

1) Find the location of the centroid of the region between the graphs of $y = e^x$ and $y = 1$ over [0, 1]. (LT: 2f)

$m = \int_0^1 \rho(e^x - 1)\,dx$

$= \rho(e^x - x)\Big|_0^1$

$= \rho(e - 1 - 1)$

$= \rho(e - 2)$

$M_x = \int_0^1 \frac{\rho}{2}(e^{2x} - 1)\,dx$

$= \frac{\rho}{2}\left(\frac{1}{2}e^{2x} - x\right)\Big|_0^1$

$= \frac{\rho}{2}\left(\frac{1}{2}e^2 - 1 - \frac{1}{2}\right)$

$= \frac{\rho}{4}(e^2 - 3)$

$M_y = \int_0^1 \rho\,(x)(e^x - 1)\,dx$

$u = x \quad dv = (e^x - 1)\,dx$

$du = dx \quad v = e^x - x$

$= \rho\left[x(e^x - x)\Big|_0^1 - \int_0^1 (e^x - x)\,dx\right]$

$= \rho\left[x(e^x - x) - e^x + \frac{x^2}{2}\right]_0^1$

$= \rho\left(e - 1 - e + \frac{1}{2} + 1\right)$

$= \frac{1}{2}\rho$

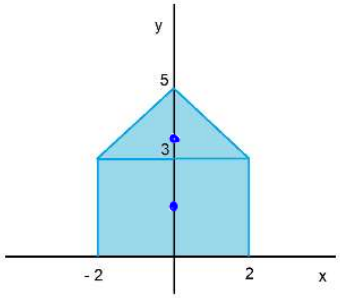
$(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right)$

$= \left(\frac{\frac{1}{2}}{e-2}, \frac{\frac{1}{4}(e^2-3)}{e-2}\right)$

$= \left(\frac{1}{2(e-2)}, \frac{e^2-3}{4(e-2)}\right)$

$(\bar{x}, \bar{y}) = \boxed{\left(\frac{1}{2(e-2)}, \frac{e^2-3}{4(e-2)}\right)}$

2) Use the additivity of moments to find the center of mass of the region shown with constant density, ρ. (LT: 2f) Use $\rho = 1$.



$m = m_T + m_R$

$m_T = \frac{1}{2}(4)(2) = 4$

$m_R = 4(3) = 12$

$m = 16$

$M_x = M_{xT} + M_{xR}$

$= \left(3 + \frac{2}{3}\right)(4) + 12\left(\frac{3}{2}\right)$

$= \frac{44}{3} + 18$

$= \frac{44 + 54}{3}$

$= \frac{98}{3}$

$=$

$M_y = M_{yT} + M_{yR}$

$= 0 + 0$

$(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right)$

$= \left(0, \frac{98}{3(16)}\right)$

$= \left(0, \frac{49}{24}\right)$

$\boxed{\left(0, \frac{49}{24}\right)}$

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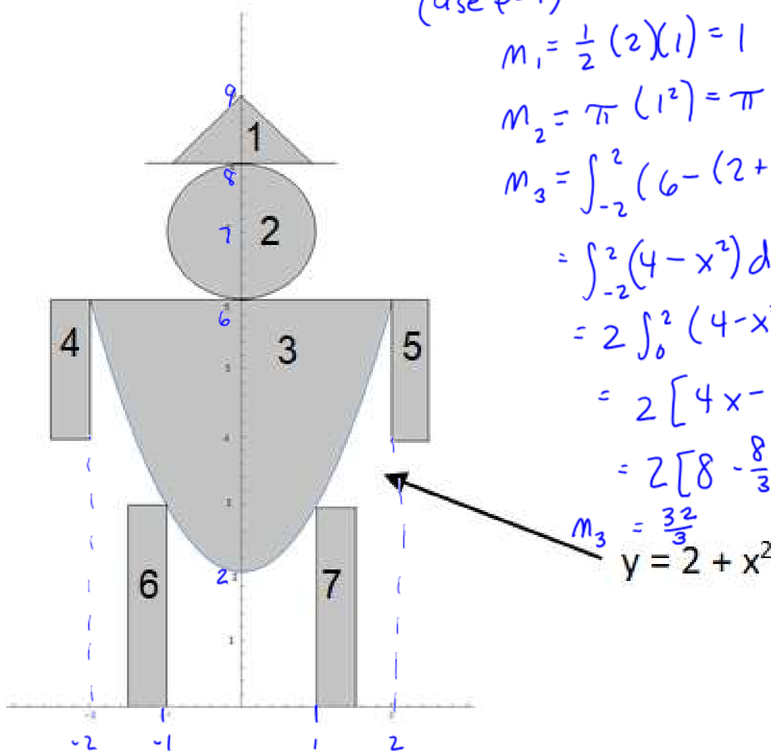
Worksheet 8.3b.2 – Moments and Center of Mass

Supplement: Find the center of mass of FlatCOMMan, if his mass density is constant. (LT: 2f)



(use $\rho = 1$)

$m_1 = \frac{1}{2}(2)(1) = 1$

$m_2 = \pi(1^2) = \pi$

$m_3 = \int_{-2}^{2} (6 - (2 + x^2))\,dx$

$= \int_{-2}^{2} (4 - x^2)\,dx$

$= 2\int_0^2 (4 - x^2)\,dx$

$= 2\left[4x - \frac{x^3}{3}\right]_0^2$

$= 2\left[8 - \frac{8}{3}\right]$

$m_3 = \frac{32}{3}$

$m_4 = m_5 = 2\left(\frac{1}{2}\right) = 1$

$m_6 = m_7 = 3\left(\frac{1}{2}\right) = \frac{3}{2}$

$m = 1 + \pi + \frac{32}{3} + 1 + 1 + \frac{3}{2} + \frac{3}{2}$

$m = \frac{50}{3} + \pi$

$M_{x1} = m_1\bar{y}_1 = 1\left(8 + \frac{1}{3}\right) = \frac{25}{3}$

$M_{x2} = m_2\bar{y}_2 = \pi(7) = 7\pi$

$M_{x3} = \int_{-2}^{2} \frac{1}{2}\left(6^2 - (2 + x^2)^2\right)dx$

$= 2\int_0^2 \frac{1}{2}\left(36 - (4 + 4x^2 + x^4)\right)dx$

$= \int_0^2 (32 - 4x^2 - x^4)\,dx$

$= \left(32x - \frac{4x^3}{3} - \frac{x^5}{5}\right)\Big]_0^2$

$= 64 - \frac{32}{3} - \frac{32}{5}$

$= \frac{960 - 160 - 96}{15}$

$M_{x3} = \frac{704}{15}$

$M_{x4} = M_{x5} = m_4\bar{y}_4 = m_5\bar{y}_5 = 1(5) = 5$

$M_{x6} = M_{x7} = m_6\bar{y}_6 = m_7\bar{y}_7 = \frac{3}{2}\left(\frac{3}{2}\right) = \frac{9}{4}$

$M_x = \frac{25}{3} + 7\pi + \frac{704}{15} + 5 + 5 + \frac{9}{4} + \frac{9}{4} = \frac{2093 + 210\pi}{30}$

$M_{y1} = M_{y2} = M_{y3} = 0$

$M_{y4} = -1\left(\frac{9}{4}\right) = -\frac{9}{4}$ $\quad M_{y5} = \frac{9}{5}$

$M_{y6} = -\frac{3}{2}\left(\frac{5}{4}\right) = -\frac{15}{8}$ $\quad M_{y7} = \frac{15}{8}$

$M_y = 0 + 0 + 0 + \left(-\frac{9}{4} + \frac{9}{4}\right) + \left(-\frac{15}{8} + \frac{15}{8}\right)$

$M_y = 0$

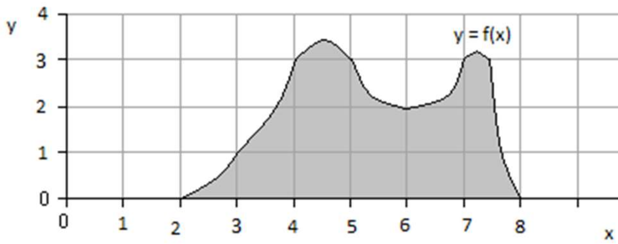
$\bar{x} = \frac{M_y}{m} = 0$

$\bar{y} = \frac{M_x}{m} = \dfrac{\frac{2093 + 210\pi}{30}}{\frac{50}{3} + \pi}$

$= \frac{2093 + 210\pi}{500 + 30\pi}$

$(\bar{x}, \bar{y}) = \boxed{\left(0, \frac{2093 + 210\pi}{500 + 30\pi}\right)}$

Supplement: Use Simpson's Rule with 6 subintervals to estimate the location of the centroid of the shaded region below. You will need to estimate three different integrals: M_x, M_y, and m.



$$S_6 = \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6\right)$$

use $\rho = 1$

$$m = \int_2^8 f(x)\,dx$$

$$m \approx \frac{1}{3}(0 + 4(1) + 2(3) + 4(3) + 2(2) + 4(3) + 0)$$

$$\approx \frac{1}{3}(4 + 6 + 12 + 4 + 12)$$

$$m \approx \frac{38}{3}$$

$$M_x = \int_2^8 \frac{1}{2}\left[f(x)\right]^2 dx$$

$$M_x \approx \frac{1}{3}\left(\frac{1}{2}\right)\left(0^2 + 4(1^2) + 2(3^2) + 4(3^2) + 2(2^2) + 4(3^2) + 0\right)$$

$$\approx \frac{1}{6}(4 + 18 + 36 + 8 + 36)$$

$$\approx \frac{102}{6}$$

$$M_x \approx 17$$

$$M_y = \int_2^8 x f(x)\,dx$$

$$M_y \approx \frac{1}{3}\left(1(2)(0) + 4(3)(1) + 2(4)(3) + 4(5)(3) + 2(6)(2) + 4(7)(3) + 1(8)(0)\right)$$

$$\approx \frac{1}{3}(12 + 24 + 60 + 24 + 84)$$

$$\approx \frac{204}{3}$$

$$M_y \approx 68$$

$$\bar{x} = \frac{M_y}{m} \approx \frac{68}{\left(\frac{38}{3}\right)} = \frac{102}{19}$$

$$\bar{y} = \frac{M_x}{m} \approx \frac{17}{\left(\frac{38}{3}\right)} = \frac{51}{38}$$

$$(\bar{x}, \bar{y}) = \boxed{\left(\frac{102}{19}, \frac{51}{38}\right)}$$