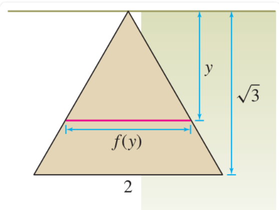
W05 - Examples

Hydrostatic force

15 - Fluid force on a triangular plate

Find the total force on the submerged *vertical* plate with the following shape: Equilateral triangle, sides 2m, top vertex at the surface, liquid is oil with density $\rho = 900\text{kg/m}^3$.



Solution

1. Establish coordinate system: height y increases going *down*.
2. Compute width function $w(y)$.
    - Drop a perpendicular from top vertex to the base.
    - Pythagorean Theorem: vertical height is $\sqrt{3}$.
    - Similar triangles: ratio $w(y)/y$ must equal ratio $2/\sqrt{3}$.
    - Solve for $w(y)$:

$$w(y) = 2y/\sqrt{3}$$

3. Write integral using width function.
    - Bounds: shallowest: $y = 0$; deepest: $y = \sqrt{3}$.
    - Integral formula:

$$F = \rho g \int_0^{\sqrt{3}} y\, w(y)\, dy = 900 \cdot 9.8 \int_0^{\sqrt{3}} y \cdot 2y/\sqrt{3}\, dy$$

4. Compute integral.
    - Simplify constants:

$$900 \cdot 9.8 \cdot \frac{2}{\sqrt{3}} \approx 10184.5$$

    - Compute integral without constants:

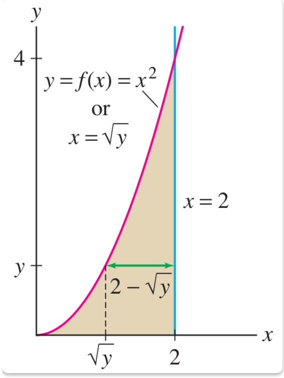
$$\int_0^{\sqrt{3}} y^2\, dy = \frac{y^3}{3}\bigg|_0^{\sqrt{3}} = \sqrt{3}$$

    - Combine for the final answer: $10184.5 \cdot \sqrt{3} = 17640$

Moments and center of mass

17 - CoM of a parabolic plate

Find the CoM of the region depicted:



Solution

1. Compute the total mass.
   - Area under the curve with density factor ρ:

$$M = \int_0^2 \rho\, x^2\, dx \quad \gg\gg \quad \rho\, \frac{x^3}{3}\bigg|_0^2 \quad \gg\gg \quad \frac{8\rho}{3}$$

2. Compute M_y.
   - Formula:

$$M_y = \int_a^b \rho\, x\, dA$$

   - Interpret and calculate:

$$M_y = \int_0^2 \rho\, x f(x)\, dx \quad \gg\gg \quad \rho \int_0^2 x^3\, dx$$

$$\gg\gg \quad 4\rho = M_y$$

3. Compute M_x.
   - Formula:

$$M_x = \int_c^d \rho\, y\, dA$$

   - Width of horizontal strips between the curves:

$$w(y) = 2 - \sqrt{y}$$

   - Interpret dA:

$$dA = (2 - \sqrt{y})\, dy$$

   - Plug data into integral:

$$M_x = \int_c^d \rho\, y\, dA \quad \gg\gg \quad \int_0^4 \rho\, y(2 - \sqrt{y})\, dy$$

   - Calculate integral:

$$\int_0^4 \rho\, y(2 - \sqrt{y})\, dy \quad \gg\gg \quad \int_0^4 \rho\, 2y\, dy - \int_0^4 \rho\, y^{3/2}\, dy$$

$$\gg\gg \quad \frac{16\rho}{5} = M_x$$

4. Compute CoM coordinates from moments.

- CoM formulas:

$$\bar{x} = \frac{M_y}{M}, \qquad \bar{y} = \frac{M_x}{M}$$

- Insert data:

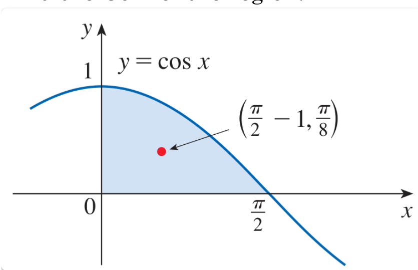
$$\bar{x} = \frac{4\rho}{8\rho/3} \quad \gg\gg \quad \frac{3}{2}$$

$$\bar{y} = \frac{16\rho/5}{8\rho/3} \quad \gg\gg \quad \frac{6}{5}$$

- Therefore CoM is located at $(\bar{x}, \bar{y}) = (\frac{3}{2}, \frac{6}{5})$.

18 - Computing CoM using only vertical strips

Find the CoM of the region:



Solution

1. Compute the total mass M.
   - Area under the curve times density ρ:

$$\int_0^{\pi/2} \rho\, \cos x\, dx = \rho \sin x \Big|_0^{\pi/2} = \rho$$

2. Compute M_y using vertical strips.
   - Plug $f(x) = \cos x$ into formula:

$$M_y = \int_a^b \rho\, x\, f(x)\, dx \quad \gg\gg \quad \int_0^{\pi/2} \rho\, x \cos x\, dx$$

   - Integration by parts.
     - Set $u = x$, $v' = \cos x$; then $u' = 1$, $v = \sin x$.
     - IBP formula:

$$\int_a^b uv'\, dx = uv\Big|_a^b - \int_a^b u'v\, dx$$

     - Plug in data:

$$\int_0^{\pi/2} \rho\, x \cos x\, dx = \rho\, x \sin x \Big|_0^{\pi/2} - \rho \int_0^{\pi/2} \sin x\, dx$$

     - Evaluate:

$$\gg\gg \qquad \frac{\pi\rho}{2} \cdot 1 - \rho\,(-\cos\frac{\pi}{2} - -\cos 0) = \rho\left(\frac{\pi}{2} - 1\right)$$

3. Compute M_x *also* using vertical strips.

- Plug $f_2(x) = f(x) = \cos x$ and $f_1(x) = 0$ into formula:

$$M_x = \int_0^{\pi/2} \rho\, \tfrac{1}{2} f_2^2\, dx \quad \gg\gg \quad \int_0^{\pi/2} \rho\, \tfrac{1}{2} \cos^2 x\, dx$$

- Integration by 'power to frequency conversion':
  - Use $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$:

$$\int_0^{\pi/2} \rho\, \tfrac{1}{2} \cos^2 x\, dx = \frac{\rho}{4} \int_0^{\pi/2} (1 + \cos 2x)\, dx$$

  - Integrate:

$$\gg\gg \qquad \frac{\rho}{4} x \Big|_0^{\pi/2} + \frac{\rho \sin 2x}{8} \Big|_0^{\pi/2} = \frac{\pi\rho}{8}$$

4. Compute CoM.
   - CoM via moment formulas:

$$\bar{x} = \frac{M_y}{M}, \qquad \bar{y} = \frac{M_x}{M}$$

   - Plug in data:

$$\bar{x} = \frac{\rho(\pi/2 - 1)}{\rho} \quad \gg\gg \quad \frac{\pi}{2} - 1$$

   - Plug in data:

$$\bar{y} = \frac{\pi\rho/8}{\rho} \quad \gg\gg \quad \frac{\pi}{8}$$

   - CoM is given by $(\bar{x}, \bar{y}) = \left(\frac{\pi}{2} - 1, \frac{\pi}{8}\right)$.

19 - CoM of region between line and parabola

Compute the CoM of the region below $y = x$ and above $y = x^2$ with $x \in [0, 1]$.

Solution

1. Name the functions: $f_1(x) = x^2$ (lower) and $f_2(x) = x$ (upper) over $x \in [0, 1]$.
2. Compute the mass M.
   - Area between curves times density:

$$M = \int_0^1 \rho\, (f_2 - f_1)\, dx \quad \gg\gg \quad \rho \int_0^1 x - x^2\, dx = \frac{\rho}{6}$$

3. Compute M_y using vertical strips.
   - Plug into formula:

$$M_y = \int_0^1 \rho\, x\, (f_2 - f_1)\, dx = \rho \int_0^1 x(x - x^2)\, dx = \frac{\rho}{12}$$

4. Compute M_x also using vertical strips.
   - Plug into formula:

$$M_x = \int_0^1 \rho\, \tfrac{1}{2} \left(f_2^2 - f_1^2\right) dx \quad \gg\gg \quad \rho \int_0^1 \tfrac{1}{2}(x^2 - x^4)\, dx = \frac{2\rho}{30}$$

5. Compute CoM.

- Using CoM via moment formulas:

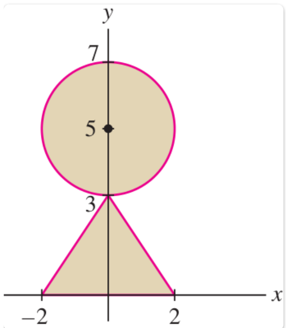
$$\bar{x} = \frac{\rho/12}{\rho/6} \quad \gg\gg \quad \frac{1}{2}$$

$$\bar{y} = \frac{2\rho/30}{\rho/6} \quad \gg\gg \quad \frac{2}{5}$$

- CoM is given by $(\bar{x}, \bar{y}) = \left(\frac{1}{2}, \frac{2}{5}\right)$.

20 - Center of mass using moments and symmetry

Find the center of mass of the region below.



Solution

1. Symmetry: CoM on y-axis.
   - Because the region is symmetric in the y-axis, the CoM must lie on that axis.
   - Therefore $\bar{x} = 0$.
2. Additivity of moments.
   - Write M_x for the total x-moment (distance measured to the x-axis from above).
   - Write M_x^{tri} and M_x^{circ} for the x-moments of the triangle and circle.
   - *Additivity of moments* equation:

$$M_x = M_x^{\text{tri}} + M_x^{\text{circ}}$$

3. Find moment of the circle M_x^{circ}.
   - By symmetry we know $\bar{x}^{\text{circ}} = 0$.
   - By symmetry we know $\bar{y}^{\text{circ}} = 5$.
   - Area of circle with $r = 2$ is $A = 4\pi$, so total mass is $M = 4\pi\rho$.
   - Centroid-from-moments equation:

$$\bar{y}^{\text{circ}} = \frac{M_x^{\text{circ}}}{M}$$

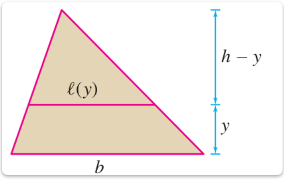
   - Solve the equation for M_x^{circ}.
     - Solve:

$$\bar{y}^{\text{circ}} = \frac{M_x^{\text{circ}}}{M} \quad \gg\gg \quad 5 = \frac{M_x^{\text{circ}}}{4\pi\rho}$$

$$\gg\gg \quad M_x^{\text{circ}} = 20\pi\rho$$

4. Find moment of the triangle M_x^{tri} using integral formula

- Similar triangles:



- Similarity equation:

$$\frac{\ell(y)}{h-y} = \frac{b}{h} \quad \gg\gg \quad \ell(y) = b - \frac{b}{h}y$$

- Integral formula:

$$M_x^{\text{tri}} = \rho \int_0^h y\ell(y)\,dy = \rho \int_0^h y\left(b - \frac{b}{h}y\right)dy$$

- Perform integral:

$$\rho \int_0^h y\left(b - \frac{b}{h}y\right)dy \quad \gg\gg \quad \rho\left(\frac{by^2}{2} - \frac{by^3}{3h}\right)\Big|_0^h = \frac{\rho b h^2}{6}$$

- Conclude:

$$M_x^{\text{tri}} = \frac{\rho b h^2}{6} = \frac{\rho 4 \cdot 3^2}{6} = 6\rho$$

5. Apply additivity.
   - Additivity formula:

$$M_x = M_x^{\text{tri}} + M_x^{\text{circ}} = \rho(20\pi + 6)$$

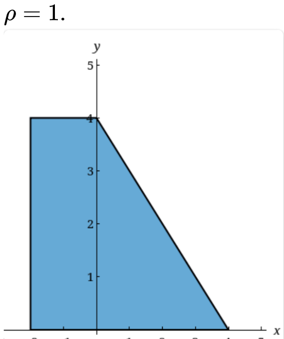
6. Total mass of region.
   - Area of circle is 4π.
   - Area of triangle is $\frac{1}{2} \cdot 4 \cdot 3 = 6$.
   - Thus $M = \rho A = \rho(4\pi + 6)$.
7. Compute center of mass $\bar{y}$ from total M_x and total M.
   - We have $M_x = \rho(20\pi + 6)$ and $M = \rho(4\pi + 6)$.
   - Plug into formula:

$$\bar{y} = \frac{M_x}{M} = \frac{\rho(20\pi + 6)}{\rho(4\pi + 6)} \approx 3.71$$

8. Final answer is $(\bar{x}, \bar{y}) = (0, 3.71)$.

21 - Center of mass - two part region

Find the center of mass of the region which combines a rectangle and triangle (as in the figure) *by computing separate moments*. What are those separate moments? Assume the mass density is $\rho = 1$.



Solution

1. ⚠ By symmetry, the center of mass of the rectangle is located at $(-1, 2)$.
   - Thus $\bar{x}^{\text{rect}} = -1$ and $\bar{y}^{\text{rect}} = 2$.
2. Find moments of the rectangle.
   - Total mass of rectangle $= M^{\text{rect}} = \rho \times \text{area} = 1 \cdot 8 = 8$.
   - Apply moment relation:

$$\bar{x}^{\text{rect}} = \frac{M_y^{\text{rect}}}{M^{\text{rect}}} \qquad \gg\gg \qquad M_y^{\text{rect}} = -8$$

$$\bar{y}^{\text{rect}} = \frac{M_x^{\text{rect}}}{M^{\text{rect}}} \qquad \gg\gg \qquad M_x^{\text{rect}} = 16$$

3. Find moments of the triangle.
   - Area of vertical slice $= \left(4 - \frac{4}{4}x\right) dx$.
   - Distance from y-axis $= x$.
   - Total M_y^{tri} integral:

$$M_y^{\text{tri}} = \int_0^4 \rho x \left(4 - \frac{4}{4}x\right) dx$$

$$= \int_0^4 1 \cdot (4 - x)x \, dx = \frac{32}{3}$$

   - Total M_x^{tri} integral:

$$M_x^{\text{tri}} = \int_0^4 \rho \frac{1}{2} f(x)^2 \, dx = \int_0^4 \rho \frac{1}{2} \left(4 - \frac{4}{4}x\right)^2 dx$$

$$= 1 \cdot \frac{1}{2} \int_0^4 (16 - 8x + x^2) \, dx = \frac{32}{3}$$

4. Add up total moments.
   - General formulas: $M_x = M_x^{\text{tri}} + M_x^{\text{rect}}$ and $M_y = M_y^{\text{tri}} + M_y^{\text{rect}}$
   - Plug in data: $M_x = \frac{32}{3} + 16 = \frac{80}{3}$ and $M_y = \frac{32}{3} - 8 = \frac{8}{3}$
5. Find center of mass from moments.
   - Total mass of triangle $= M^{\text{tri}} = \rho \times \text{area} = 1 \cdot \frac{1}{2} \cdot 4 \cdot 4 = 8$.
   - Total combined mass $= M = M^{\text{tri}} + M^{\text{rect}} = 8 + 8 = 16$.
   - Apply moment relation:

$$\bar{x} = \frac{M_y}{M} = \frac{8/3}{16} = \frac{1}{6}$$

$$\bar{y} = \frac{M_x}{M} = \frac{80/3}{16} = \frac{5}{3}$$

   - Therefore, center of mass is $(\bar{x}, \bar{y}) = \left(\frac{1}{6}, \frac{5}{3}\right)$.