Name: Solutions

Worksheet 8.1 – Arc Length

1) Find the arc length of the curve $y = 3x + 1$ over the interval $0 \le x \le 3$. (LT 2d)

$$\ell = \int_a^b \sqrt{1 + (dy/dx)^2}\, dx \qquad \frac{dy}{dx} = 3$$

$$\ell = \int_0^3 \sqrt{1+9}\, dx$$

$$= \sqrt{10}\, dx \Big|_0^3$$

$$= 3\sqrt{10}$$

$$\boxed{3\sqrt{10}}$$

2) Find the arc length of the curve $x = \frac{1}{12}y^3 + y^{-1}$ over the interval $1 \le y \le 2$. (LT: 2d)

$$\ell = \int_c^d \sqrt{1 + (dx/dy)^2}\, dy \quad \longrightarrow$$

$$\frac{dx}{dy} = \frac{1}{4}y^2 - \frac{1}{y^2}$$

$$\left(\frac{dx}{dy}\right)^2 = \frac{1}{16}y^4 - \frac{2}{4}\frac{y^2}{y^2} + \frac{1}{y^4}$$

$$= \frac{1}{16}y^4 - \frac{1}{2} + \frac{1}{y^4}$$

$$1 + \left(\frac{dx}{dy}\right)^2 = \frac{1}{16}y^4 + \frac{1}{2} + \frac{1}{y^4} = \frac{y^8 + 8y^4 + 16}{16y^4}$$

$$\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{\frac{(y^4+4)^2}{(4y^2)^2}} = \frac{y^4+4}{4y^2}$$

$$\ell = \int_1^2 \frac{y^4+4}{4y^2}\, dy$$

$$= \int_1^2 \left(\frac{y^2}{4} + \frac{1}{y^2}\right) dy$$

$$= \left(\frac{y^3}{12} - \frac{1}{y}\right)\Big|_1^2$$

$$= \frac{8}{12} - \frac{1}{2} - \frac{1}{12} + 1$$

$$= \frac{8 - 6 - 1 + 12}{12}$$

$$= \frac{13}{12}$$

$$\boxed{\frac{13}{12}}$$

Supplement: Find the arc length of the curve $y = e^x$ over $\left[0, \frac{1}{2}\right]$. (LT: 2d)

$$L = \int_0^{\frac{1}{2}} \sqrt{1+e^{2x}}\, dx$$

option 1:

$u = e^x \quad du = e^x dx \to dx = \frac{1}{u} du$

$$L = \int_1^{\sqrt{e}} \frac{\sqrt{1+u^2}}{u}\, du$$

$u = \tan\theta \quad du = \sec^2\theta\, d\theta$

$$L = \int_{u=1}^{u=\sqrt{e}} \frac{\sqrt{1+\tan^2\theta}}{\tan\theta} \sec^2\theta\, d\theta$$

$$= \int_{u=1}^{u=\sqrt{e}} \frac{\sec\theta}{\tan\theta}(1+\tan^2\theta)\, d\theta$$

$$= \int_{u=1}^{u=\sqrt{e}} (\csc\theta + \tan\theta\sec\theta)\, d\theta$$

$$= \left(\ln|\csc\theta - \cot\theta| + \sec\theta\right)\Big|_{u=1}^{u=\sqrt{e}}$$


$$= \left(\ln\left|\frac{\sqrt{u^2+1}}{u} - \frac{1}{u}\right| + \sqrt{u^2+1}\right)\Big|_1^{\sqrt{e}}$$

$$= \ln\left|\frac{\sqrt{e+1}}{\sqrt{e}} - \frac{1}{\sqrt{e}}\right| + \sqrt{e+1} - \ln|\sqrt{2}-1| - \sqrt{2}$$

$$= \ln(\sqrt{e+1}-1) - \frac{1}{2} + \sqrt{e+1} - \ln(\sqrt{2}-1) - \sqrt{2}$$

$\approx 0.821028$

option 2:

$u = \sqrt{1+e^{2x}}$

$u^2 = 1+e^{2x} \to e^{2x} = u^2-1$

$2u\,du = 2e^{2x}dx \to dx = \frac{u}{u^2-1}du$

$$L = \int_{\sqrt{2}}^{\sqrt{1+e}} u\left(\frac{u}{u^2-1}\right) du$$

$$= \int_{\sqrt{2}}^{\sqrt{1+e}} \left(1 + \frac{1}{u^2-1}\right) du$$

$\frac{1}{u^2-1} = \frac{A}{u-1} + \frac{B}{u+1}$

$1 = A(u+1) + B(u-1)$

$u=-1 \to \quad 1 = -2B \to B = -\frac{1}{2}$

$u=1 \to \quad 1 = 2A \to A = \frac{1}{2}$

$$= \int_{\sqrt{2}}^{\sqrt{1+e}} \left[1 + \frac{1}{2}\left(\frac{1}{u-1}\right) - \frac{1}{2}\left(\frac{1}{u+1}\right)\right] du$$

$$= \left(u + \frac{1}{2}\ln|u-1| - \frac{1}{2}\ln|u+1|\right)\Big|_{\sqrt{2}}^{\sqrt{e+1}}$$

$$= \sqrt{e+1} + \frac{1}{2}\ln|\sqrt{e+1}-1| - \frac{1}{2}\ln|\sqrt{e+1}+1| - \sqrt{2} - \frac{1}{2}\ln|\sqrt{2}-1| + \frac{1}{2}\ln(\sqrt{2}+1)$$

$\approx 0.821028$

$$\boxed{\ln(\sqrt{e+1}-1) - \frac{1}{2} + \sqrt{e+1} - \ln(\sqrt{2}-1) - \sqrt{2}}$$

Name: Solutions

Worksheet 8.2 – Area of Surface of Revolution

1) Find the surface area of revolution about the x-axis of the curve $y = x^3$ for $0 \le x \le 2$. (LT: 2e)

$S = \int 2\pi y\, ds \rightarrow y = x^3 \quad ds = \sqrt{1 + (dy/dx)^2}\, dx$

$= \sqrt{1 + (3x^2)^2}\, dx$

$= \sqrt{1 + 9x^4}\, dx$

$S = \int_0^2 2\pi x^3 \sqrt{1 + 9x^4}\, dx$

$u = 1 + 9x^4$

$du = 36x^3\, dx$

$S = \int_1^{145} \frac{2\pi}{36} \sqrt{u}\, du$

$= \frac{\pi}{18}\left(\frac{2}{3}\right)\left(u^{3/2}\right)\Big|_1^{145}$

$= \frac{\pi}{27}\left(145^{3/2} - 1^{3/2}\right)$

$\boxed{\frac{\pi}{27}\left(145\sqrt{145} - 1\right)}$

2) Set up an integral to find the surface area of revolution about the y-axis of the curve $y = x^3$ for $0 \le x \le 2$. Do not evaluate. (LT: 2e)

The only thing that changes is the radius of revolution.

$S = \int 2\pi x\, ds$

$\boxed{S = \int_0^2 2\pi x \sqrt{1 + 9x^4}\, dx}$

3) Find the area of the surface of revolution about the x-axis of the curve $x = \frac{1}{12}y^3 + y^{-1}$ over the interval $1 \le y \le 2$. You may reuse your work from Worksheet 8.1. (LT: 2e)

$S = \int 2\pi r\, ds$

From worksheet 8.1a, $ds = \left(\frac{y^2}{4} + \frac{1}{y^2}\right) dy$

$r = y$

$S = \int_1^2 2\pi y \left(\frac{y^2}{4} + \frac{1}{y^2}\right) dy$

$= \int_1^2 2\pi \left(\frac{y^3}{4} + \frac{1}{y}\right) dy$

$= 2\pi \left(\frac{y^4}{16} + \ln|y|\right)\Big|_1^2$

$= 2\pi \left(\frac{16}{16} + \ln 2 - \frac{1}{16} - \ln 1\right)$

$= 2\pi \left(\frac{15}{16} + \ln 2\right)$

$\boxed{2\pi \left(\frac{15}{16} + \ln 2\right)}$