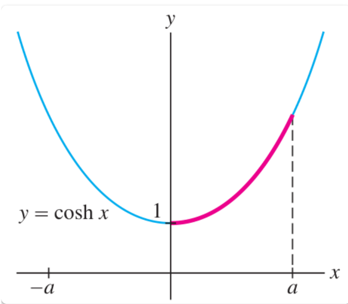
W04 - Examples

Arc length

12 - Arc length of chain, via position

A hanging chain describes a **catenary** shape. ('Catenary' is to hyperbolic trig as 'sinusoid' is to normal trig.) The graph of the hyperbolic cosine is a catenary:

$$y = f(x) = \cosh x$$



Let us compute the arc length of this catenary on the portion from $x = 0$ to $x = a$.

Solution

1. ⚠ Arc-length formula.
   - Give arc length $s(a)$, a function of $a \geq 0$:

$$s(a) = \int_0^a \sqrt{1 + (f')^2}\, dx$$

2. ≡ Compute $f'(x)$.
   - Hyperbolic trig derivative:

$$\frac{d}{dx}\cosh(x) = \sinh(x)$$

3. ≡ Plug into formula.
   - Arc length:

$$s(a) = \int_0^a \sqrt{1 + \sinh^2(x)}\, dx$$

4. Hyperbolic trig identity.
   - Fundamental identity:

$$\cosh^2 x - \sinh^2 x = 1$$

   - Rearrange:

$$1 + \sinh^2 x = \cosh^2 x$$

5. Plug into formula and compute.
   - Arc length:

$$\int_0^a \sqrt{1 + \sinh^2(x)}\, dx \quad \gg\gg \quad \int_0^a \sqrt{\cosh^2 x}\, dx \quad \gg\gg \quad \int_0^a \cosh x\, dx$$

- Compute integral:

$$\int_0^a \cosh x \, dx = \sinh a$$

- ⓘ The arc length of a catenary curve matches the area under the catenary curve!

13 - Arc length, line segment

Find the arc length of the *straight line* given by the formula $y = 3x + 1$ over $x \in [0, 3]$.

Check your answer using the Pythagorean Theorem.

Surface areas of revolutions - thin bands

14 - Surface area of a sphere

Using the fact that a sphere is given by revolving a semicircle, verify the formula $S = 4\pi r^2$ for the surface area of a sphere.

Solution

1. Sphere as surface of revolution.
    - Sphere of radius r given by revolving upper semicircle.
    - Upper semicircle:

    $$x^2 + y^2 = r^2, \qquad y \geq 0$$

    - Upper semicircle as function of x:

    $$y = f(x) = \sqrt{r^2 - x^2}, \qquad x \in [-r, r]$$

2. Surface area formula.
    - Bounds are $x = -r$ and $x = +r$.
    - Function is $f(x) = \sqrt{r^2 - x^2}$
    - Plug data into formula:

    $$S = \int_{-r}^{+r} 2\pi \sqrt{r^2 - x^2} \sqrt{1 + \left(f'\right)^2} \, dx$$

3. Compute $\left(f'\right)^2$.
    - Power rule and chain rule:

    $$f'(x) = \frac{1}{2}(r^2 - x^2)^{-1/2}(-2x)$$

    - Algebra:

    $$\gg\gg \qquad -x(r^2 - x^2)^{-1/2}$$

    - Squaring:

    $$\left(f'\right)^2 = \frac{x^2}{r^2 - x^2}$$

4. Compute integrand.

- Compute $1 + \left(f'\right)^2$:

$$1 = \frac{r^2 - x^2}{r^2 - x^2}$$

$$1 + \left(f'\right)^2 \qquad \gg\gg \qquad \frac{r^2 - x^2}{r^2 - x^2} + \frac{x^2}{r^2 - x^2} \qquad \gg\gg \qquad \frac{r^2}{r^2 - x^2}$$

- Integrand factors become:

$$\sqrt{r^2 - x^2}\sqrt{1 + \left(f'\right)^2} \qquad \gg\gg \qquad \sqrt{r^2 - x^2}\sqrt{\frac{r^2}{r^2 - x^2}} \qquad \gg\gg \qquad r$$

5. Compute integral.
   - Surface area again:

$$S = \int_{-r}^{+r} 2\pi r \, dx$$

$$= 2\pi r x \Big|_{-r}^{+r} = 2\pi r r - 2\pi r(-r)$$
$$= 4\pi r^2$$

   - This is the desired surface area formula $S = 4\pi r^2$.