Name: Solutions

Worksheet 7.4a – Integration of Rational Functions by Partial Fractions

1) $\int \frac{x^2}{x^2+9}\,dx =$

Two ways to divide:

① $\frac{x^2}{x^2+9} = \frac{x^2+9-9}{x^2+9} = 1 - \frac{9}{x^2+9}$ OR ② $x^2+9 \overline{)\,x^2}$ gives $1 - \frac{9}{x^2+9}$: $x^2 - (x^2+9) = -9$

$$\int \frac{x^2}{x^2+9}\,dx = \int \left(1 - \frac{9}{x^2+9}\right) dx$$

$$= x - \frac{9}{3}\tan^{-1}\frac{x}{3} + C$$

$$\boxed{x - 3\tan^{-1}\frac{x}{3} + C}$$

2) $\int \frac{(x^2-x+1)}{x^2+x}\,dx =$

$x^2+x \overline{)\,x^2-x+1}$ gives 1: $(x^2-x+1) - (x^2+x) = -2x+1$

$$\frac{-2x+1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

$$-2x+1 = A(x+1) + Bx$$

$x=0 \rightarrow 1 = A$

$x=-1 \rightarrow 3 = -B \rightarrow B = -3$

$$\int \frac{x^2-x+1}{x^2+x}\,dx = \int \left[1 + \frac{-2x+1}{x^2+x}\right] dx$$

$$= x + \int \frac{-2x+1}{x(x+1)}\,dx$$

$$= x + \int \left(\frac{1}{x} - \frac{3}{x+1}\right) dx$$

$$= x + \ln|x| - 3\ln|x+1| + C$$

$$\boxed{x + \ln|x| - 3\ln|x+1| + C}$$

3) $\int \frac{5x^2-5x+14}{(x-2)(x^2+4)}dx =$

$$\frac{5x^2-5x+14}{(x-2)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+4}$$

$$5x^2-5x+14 = A(x^2+4) + (Bx+C)(x-2)$$

$$x=2 \rightarrow 24 = 8A \rightarrow A=3$$

$$x=0 \rightarrow 14 = 4A - 2C \rightarrow 14 = 12-2C \rightarrow C=-1$$

$$5x^2-5x+14 = Ax^2+4A+Bx^2+Cx-2Bx-2C$$

$$5x^2-5x+14 = x^2(A+B)+x(C-2B)+(4A-2C) \quad \text{(match coefficients)}$$

$$A+B=5 \rightarrow B=2$$

$$\int \frac{5x^2-5x+14}{(x-2)(x^2+4)}dx = \int\left(\frac{3}{x-2}+\frac{2x-1}{x^2+4}\right)dx$$

$$= \int\left(\frac{3}{x-2}+\frac{2x}{x^2+4}-\frac{1}{x^2+4}\right)dx$$

$$= 3\ln|x-2| + \frac{2}{2}\ln(x^2+4) - \frac{1}{2}\tan^{-1}\frac{x}{2} + C$$

$$\boxed{3\ln|x-2| + \ln(x^2+4) - \frac{1}{2}\tan^{-1}\frac{x}{2} + C}$$

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Worksheet 7.4b – Integration of Rational Functions by Partial Fractions

1) What is the best partial fraction decomposition of $\dfrac{x+2}{(x^2+2)(x-1)^3(x^2-9)}$? (Show the decomposition with the undetermined coefficients. Don't solve for the coefficients.) (LT: 1g)

$$\boxed{\frac{Ax+B}{x^2+2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3} + \frac{F}{x-3} + \frac{G}{x+3}}$$

For problems 2, 3, and 4, solve the integral. You might need to make a substitution first before decomposing into partial fractions. (LT: 1g, 1h)

2) $\displaystyle\int \frac{(x^2+11x)}{(x-1)(x+1)^2}\,dx$

$$\frac{x^2+11x}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$x^2+11x = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$$

$x=1 \rightarrow 12 = 4A \rightarrow A=3$

$x=-1 \rightarrow -10 = -2C \rightarrow C=5$

$x^2+11x = x^2(A+B) + x(2A+C) + (A-B-C)$

$A+B=1 \rightarrow B=-2$

$2A+C=11?$ ✓

$A-B-C=0?$ ✓

$$\int \frac{x^2+11x}{(x-1)(x+1)^2}\,dx = \int\left(\frac{3}{x-1} + \frac{-2}{x+1} + \frac{5}{(x+1)^2}\right)dx$$

$$= 3\ln|x-1| - 2\ln|x+1| - 5(x+1)^{-1} + C$$

$$\boxed{3\ln|x-1| - 2\ln|x+1| - \frac{5}{x+1} + C}$$

3) $\int \frac{1}{e^x - 1}\,dx$

$u = e^x$

$du = e^x\,dx$

$\int \frac{1}{e^x-1}\,dx = \int \frac{e^x}{e^x(e^x-1)}\,dx$

$= \int \frac{du}{u(u-1)}$

$= \int \left(-\frac{1}{u} + \frac{1}{u-1}\right)du$

$= -\ln|u| + \ln|u-1| + C$

$= -\ln(e^x) + \ln|e^x-1| + C$

$= -x + \ln|e^x-1| + C$

$\frac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1}$

$1 = A(u-1) + Bu$

$u = 1 \rightarrow 1 = B$

$u = 0 \rightarrow 1 = -A \rightarrow A = -1$

$$\boxed{-x + \ln|e^x - 1| + C}$$

4) $\int \frac{\sqrt{x}}{x-1}\,dx$

$u = \sqrt{x}$

$u^2 = x$

$2u\,du = dx$

$\int \frac{\sqrt{x}}{x-1}\,dx = \int \frac{u}{u^2-1}(2u)\,du$

$= \int \frac{2u^2}{u^2-1}\,du$

$= \int \left(2 + \frac{2}{u^2-1}\right)du$

$= \int \left(2 + \frac{1}{u-1} - \frac{1}{u+1}\right)du$

$= 2u + \ln|u-1| - \ln|u+1| + C$

$= 2\sqrt{x} + \ln|\sqrt{x}-1| - \ln(\sqrt{x}+1) + C$

$$\begin{array}{r} 2 \\ u^2-1 \enclose{longdiv}{2u^2} \\ \underline{2u^2-2} \\ 2 \end{array}$$

$\frac{2}{u^2-1} = \frac{A}{u-1} + \frac{B}{u+1}$

$2 = A(u+1) + B(u-1)$

$u = -1 \rightarrow 2 = -2B \rightarrow B = -1$

$u = 1 \rightarrow 2 = 2A \rightarrow A = 1$

$$\boxed{2\sqrt{x} + \ln|\sqrt{x}-1| - \ln(\sqrt{x}+1) + C}$$

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Worksheet 7.7 – Approximate Integration

1) a) Calculate S_6 for $\int_0^3 x^5\,dx$. Then compute A, the actual value of the integral. Then compute E_S. You may use a calculator. (LT: 1i)

$S_6 = \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6)$ where $y_i = x_i^5$

$\Delta x = \frac{3-0}{6} = \frac{1}{2}$

$S_6 = \frac{\left(\frac{1}{2}\right)}{3}\left(0^5 + 4\left(\frac{1}{2}\right)^5 + 2(1)^5 + 4\left(\frac{3}{2}\right)^5 + 2(2)^5 + 4\left(\frac{5}{2}\right)^5 + 3^5\right)$

$= 121.6875$

$A = \int_0^3 x^5\,dx$

$= \frac{x^6}{6}\Big|_0^3$

$= \frac{3^6}{6}$

$= 121.5$

$E = A - S_6$

$= 121.5 - 121.6875$

$= -0.1875$

$S_6 = 121.6875$	$A = 121.5$	$E_S = -0.1875$

b) Find the smallest value of n for which $Error(S_n) \le 10^{-9}$ for the integral, $\int_0^3 x^5 dx$. Given: $|E_S| \le \frac{K(b-a)^5}{180n^4}$. You may use a calculator. (LT: 1j)

$f(x) = x^5$

$f'(x) = 5x^4$

$f''(x) = 20x^3$

$f^{(3)}(x) = 60x^2$

$f^{(4)}(x) = 120x$

$K = 360$

$|E_s| \le \frac{360(3-0)^5}{180n^4} \le 10^{-9}$

$n^4 \ge 4.86 \times 10^{11}$

$n \ge 834.947$

$n \ge 836$ (must be an even integer)

$\boxed{836}$

2) Use Simpson's Rule to estimate the average temperature in a museum over a 3-hour period, if the temperature (in degrees Celsius), recorded at 15-minute intervals, are

Elapsed Time (min)	0	15	30	45	60	75	90	105	120	135	150	165	180
Temp (ºC)	21	21.3	21.5	21.8	21.6	21.2	20.8	20.6	20.9	21.2	21.1	21.3	21.2

You may use a calculator. (LT: 1i)

Hint: $f_{AVE} = \frac{1}{b-a}\int_a^b f(x)\,dx$

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\left(y_0+4y_1+2y_2+4y_3+2y_4+4y_5+2y_6+4y_7+2y_8+4y_9+2y_{10} + 4y_{11} + y_{12}\right)$$

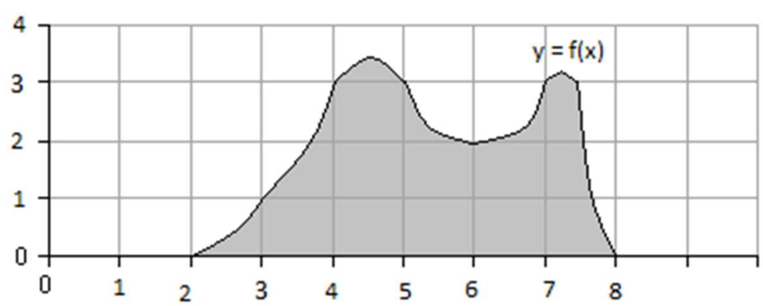
$$\approx \frac{15}{3}\left(21+4(21.3)+2(21.5)+4(21.8)+2(21.6)+4(21.2)+2(20.8) + 4(20.6)+2(20.9)+4(21.2)+2(21.1)+4(21.3)+21.2\right)$$

$$\approx 3818$$

$$f_{ave} \approx \frac{1}{180-0}(3818) = 21.211\,°C$$

$\boxed{21.211°C}$

3) Use S_6 to approximate the volume of the solid region obtained by revolving the shaded plane region below *f(x)* around the *y-axis*. Try not to use a calculator. (LT: 1i)



$$V = \int_2^8 2\pi x f(x)\,dx$$

$$V \approx S_6$$

$$\approx \frac{\Delta x}{3}\left(y_0+4y_1+2y_2+4y_3+2y_4+4y_5+y_6\right)$$

where $y_i = 2\pi x_i f(x_i)$

and $\Delta x = \frac{8-2}{6} = 1$

$$V \approx \frac{1}{3}(2\pi)\left(2(0)+4(3)(1)+2(4)(3)+4(5)(3)+2(6)(2)+4(7)(3)+8(0)\right)$$

$$\approx \frac{2\pi}{3}(204)$$

$$\approx 136\pi$$

$$\approx 427.257$$

$\boxed{136\pi}$